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An integral-type constraint qualification to guarantee nondegeneracy of the maximum principle for optimal control problems with state constraints[☆]

S.O. Lopes [a], F.A.C.C. Fontes [b,*], M.d.R. de Pinho [b]

[a] CMAT and Department of Mathematics and Applications, University of Minho, 4800-058 Guimarães, Portugal
[b] ISR-Porto, Faculdade de Engenharia, Universidade do Porto, 4200-465 Porto, Portugal

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ABSTRACT

For optimal control problems involving ordinary differential equations and functional inequality state constraints, the maximum principle may *degenerate*, producing no useful information about minimizers. This is known as the degeneracy phenomenon. Several non-degenerate forms of the maximum principle, valid under different constraint qualifications, have been proposed in the literature.

In this paper we propose a new constraint qualification under which a nondegenerate maximum principle is validated. In contrast with existing results, our constraint qualification is of an integral type. An advantage of the proposed constraint qualification is that it is verified on a larger class of problems with nonsmooth data and convex velocity sets.

1. Introduction

Since the birth of optimal control theory, commonly assumed to be in the late 50's of last century [1], the maximum principle (MP) has been a powerful and widely used analytic tool. As it is well known, the maximum principle provides a set of necessary optimality conditions useful to identify, among admissible solutions, candidates to minimizers. The original statement of the maximum principle presented by Pontryagin et al. has been generalized, strengthened and extended in many different ways. A major driving force behind these and other developments in optimal control theory has been the increasing number of applications.

Since state constraints are repeatedly encountered in applications, it is no surprise that the state constrained maximum principles have been the focus of intense research. Particularly relevant for our context is the work of Dubovitskii and Milyutin [2], which introduced measures in the maximum principle for such problems, and its extension to nonsmooth problems by Vinter and Pappas [3]. The state constrained maximum principle may exhibit a troublesome shortcoming. Indeed, and as it is amply illustrated by an example by Dubovitskii (see description in [4] and references therein), the maximum principle may degenerate if one end of the optimal trajectory belongs to the boundary of the state constraints. This phenomenon is known in the literature as the degeneracy phenomenon of the maximum principle for state constrained problems. It may arise in applications, most notably when Model Predictive Control frameworks are used (see e.g. [5] for a description of this technique) since the optimal control problems have to be solved for several initial states along the trajectory.

Clearly the occurrence of the degeneracy phenomenon has as a consequence that the necessary optimality conditions no longer give useful information to select minimizers. To remedy such a situation several authors have come up with conditions designed to identify classes of problems for which the maximum principle is nondegenerate (see, for example, [6–13], etc.). Other situations that may be related to the degeneracy phenomenon as *normality* and *regularity* of the optimal control and multipliers have also been amply studied; in this respect see [14–23] among others.

In the literature, the conditions imposed to avoid the degeneracy phenomenon, called *constraint qualifications*, are *inward pointing type conditions* of mainly two types (see [13] for a discussion). One type of these conditions assumes knowledge of the optimal

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* Corresponding author. Tel.: +351 936196325.
E-mail addresses: sofialopes@math.uminho.pt (S.O. Lopes), faf@fe.up.pt, fontes.fa@gmail.com (F.A.C.C. Fontes), mrpinho@fe.up.pt (M.d.R. de Pinho).

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control. Although this type of conditions holds under less regularity assumptions, it has the disadvantage of being difficult to verify since the optimal control is not known a priori.

In this paper we focus on necessary conditions of optimality for state constrained problems. We propose a new and weaker type of inward pointing conditions to avoid the occurrence of the degeneracy phenomenon of the state constrained maximum principle. Differing from the literature, our constraint qualification is of integral-type (a preliminary version of these results was announced in [24,25]).

Our constraint qualification is a condition that implies, but is not implied by the constraint qualification in [9]. The accompanying nondegenerate maximum principle applies to problems with possibly nonsmooth data. The price we pay is that convexity of the so-called "velocity set" is assumed. Therefore, the results proposed here, can be applied to a larger class of problems with nonsmooth data and convex velocity sets.

This paper is organized as follows. We start by giving the main concepts and notation that are used throughout the paper in the next section. In Section 2 we describe, in the context of our results, optimal control problems with state constraints, the maximum principle, the degeneracy phenomenon and the literature on constraints qualifications designed to avoid the degeneracy phenomenon. Our integral type constraint qualification as well as the statement of the associated nondegenerate maximum principle is introduced and discussed in Section 3 where we state a smooth version of our main result. Section 4 focuses on the nonsmooth more general case. The proof of our main result is in Section 5.

2. Necessary conditions of optimality and the degeneracy phenomenon

Consider an optimal control problem with fixed initial state and with pathwise constraints:

$$(P)\begin{cases}\text{Minimize } g(x(1)) & \\ \text{subject to } \dot{x}(t)=f\,(t,x(t),u(t)) & \text{a.e. } t\in[0,1]\\ x(0)=x_0 & \\ x(1)\in C & \\ u(t)\in\Omega(t) & \text{a.e. } t\in[0,1]\\ h\,(t,x(t))\le 0 & \text{for all } t\in[0,1].\end{cases}$$

The data for this problem comprises functions $g:\mathbb{R}^n\mapsto\mathbb{R}, f:\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^m\mapsto\mathbb{R}^n, h:\mathbb{R}\times\mathbb{R}^n\mapsto\mathbb{R}$, and a multifunction $\Omega:[0,1]\rightrightarrows\mathbb{R}^m$. The set of *control functions* for (P) is $\mathcal{U}:=\{u:[0,1]\mapsto\mathbb{R}^m : u \text{ is a measurable function}, u(t)\in\Omega(t) \text{ a.e. } t\in[0,1]\}$. The *state trajectory* is an absolutely continuous function which satisfies the differential equation for some control function u. The domain of the above optimization problem is the set of *admissible processes*, namely pairs (x,u) comprising a control function u and a corresponding state trajectory x which satisfy the constraints of (P). We say that an admissible process $(\bar{x},\bar{u})$ is a *local minimizer* if there exists $\delta>0$ such that $g(\bar{x}(1))\le g(x(1))$ for all admissible processes (x,u) satisfying $\|x-\bar{x}\|_{L^\infty}\le\delta$.

The MP for problems with state constraints, featuring measures as the multipliers associated with the such constraints, were first introduced by Dubovitskii and Milyutin in [2]. Several generalizations were developed, see for example [26–28].

Assume that, for some $\delta'>0$, the following hypotheses are satisfied.

H1. The function $(t,u)\mapsto f(t,x,u)$ is $\mathcal{L}\times\mathcal{B}^m$ measurable for each x.

H2. There exists a $\mathcal{L}\times\mathcal{B}^m$ measurable function $k(t,u)$ such that $t\mapsto k(t,\bar{u}(t))$ is integrable and

$$\|f(t,x,u)-f(t,x',u)\|\le k(t,u)\|x-x'\|$$

for $x,x'\in\bar{x}(t)+\delta'\mathbb{B}, u\in\Omega(t)$ a.e. $t\in[0,1]$. There exist scalars $K_f>0$ and $\epsilon'>0$ such that

$$\|f(t,x,u)-f(t,x',u)\|\le K_f\|x-x'\|,$$

for $x,x'\in\bar{x}(0)+\delta'\mathbb{B}, u\in\Omega(t)$ a.e. $t\in[0,\epsilon']$.

H3. The function g is Lipschitz continuous on $\bar{x}(1)+\delta'\mathbb{B}$.

H4. The graph of Ω is $\mathcal{L}\times\mathcal{B}^m$ measurable.

H5. The set C is closed.

H6. The function h is upper semicontinuous in t and there exists a scalar $K_h>0$ such that

$$|h(t,x)-h(t,x')|\le K_h\|x-x'\|,$$

for all $t\in[0,1]$.

H7. There exist positive constants ϵ and ϵ_1 such that $f(t,x,\Omega(t))$ is convex for all $t\in[0,\epsilon)$ and for all $x\in x_0+\epsilon_1\mathbb{B}$.

Here $\mathbb{B}$ denotes the *closed unit ball* and $\mathcal{L}\times\mathcal{B}^m$ denotes the product σ-algebra generated by Lebesgue sets of [0, 1] and Borel subsets of $\mathbb{R}^m$.

To simplify the exposition we start to present in this section and in Section 3 the smooth case. Thus we add the following additional interim hypotheses:

AH2. The function $x\mapsto f(t,x,u)$ is continuously differentiable for each (t,u).

AH3. The function g is continuous differentiable on $\bar{x}(1)+\delta'\mathbb{B}$.

AH5. The set C is convex.

AH6. The function $x\mapsto h(t,x)$ is differentiable for fixed t and h and h_x are continuous.

These hypotheses will be removed later in the main result, in Section 4.

The maximum principle (MP) for state constraints typically asserts existence of an absolutely continuous function p, a nonnegative regular Borel measure $\mu\in C^*([0,1],\mathbb{R})$, and a scalar $\lambda\ge 0$ satisfying

$$\mu\{[0,1]\}+\|p\|_{L^\infty}+\lambda>0, \tag{1}$$

$$-\dot{p}(t)=\left(p(t)+\int_{[0,t)}h_x(s,\bar{x}(s))\mu(ds)\right)\cdot f_x\,(t,\bar{x}(t),\bar{u}(t)) \quad \text{a.e. } t\in[0,1], \tag{2}$$

$$-\left(p(1)+\int_{[0,1]}h_x(s,\bar{x}(s))\mu(ds)\right)\in N_C(\bar{x}(1))+\lambda g_x(\bar{x}(1)), \tag{3}$$

$$\text{supp}\,\{\mu\}\subset\{t\in[0,1]:h\,(t,\bar{x}(t))=0\}, \tag{4}$$

and for almost every $t\in[0,1]$, $\bar{u}(t)$ maximizes over $\Omega(t)$

$$u\mapsto\left(p(t)+\int_{[0,t)}h_x(s,\bar{x}(s))\mu(ds)\right)\cdot f\,(t,\bar{x}(t),u), \tag{5}$$

where supp $\{\mu\}$ denotes the support of measure μ, C^* the dual space to the space of continuous functions and $N_C(x)$ denotes the normal cone to C at x. (For convex sets the normal cone is simply $N_C(x):=\{y\in\mathbb{R}^n : y^T(x'-x)\le 0, x'\in C\}$. For nonconvex sets it will be defined later in Section 4.)

This MP might not supply any useful information to select minimizers for certain optimal control problems where the trajectory starts on the boundary of the admissible state region, i.e.,