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Feedback linearization and lattice theory[☆,☆☆]

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ABSTRACT

The tools of lattice theory are applied to readdress the static state feedback linearization problem for discrete-time nonlinear control systems. Unlike the earlier results that are based on differential geometry, the new tools are also applicable for nonsmooth systems. In case of analytic systems, close connections are established between the new results and those based on differential one-forms. The Mathematica functions have been developed that implement the algorithms/methods from this paper.

1. Introduction

The paper recasts the old problem of static state feedback linearization using the algebraic lattice theory. The mathematical technique used is known under the name 'algebra of functions' [1]. The interest in recasting the old problem is manyfold. First, it helps through comparison to evaluate the (dis)advantages of the new technique. The explicit relations between respective formulas/algorithms/solvability conditions can be given. Second, it allows to compare the assumptions behind the different approaches. Finally, since the new tools were inspired by the algebra of partitions in the theory of finite automata [2] and mimics the latter, it helps to build a possible bridge between the theories of continuous-time and discrete event systems. However, this aspect will be studied in another paper.

In the algebra of functions the partitions (used in the algebra of partitions) were replaced by functions generating them and the analogous operations and operators for functions were introduced. The four key elements of the algebra of functions are partial preorder relation, binary operations (sum and product, defined in a specific manner), binary relation and certain operators **m** and **M**, defined on the set S_X of vector functions with the domain being the state space X of the control system. The starting point of the approach are the relations of partial preorder and equivalence, denoted as $\cong$. The equivalence relation divides the set S_X of vector functions into the equivalence classes $S_X \setminus \cong$ which is proved to be a lattice. Like the tools based on the differential forms, the algebra of functions provides a unified viewpoint to study the discrete-time as well as the continuous-time control systems; additionally it allows to address also the discrete-event systems like those in [3,4]. An important point to stress is that these tools (unlike most previous methods) do not require the system to be described in terms of smooth functions.

Besides extending the results on feedback linearization for nonsmooth systems, the goal of this paper is to compare the tools of the algebra of functions with those based on the vector spaces of differential one-forms over suitable difference fields of nonlinear functions. We will give precise relations between respective solvability conditions and solutions for analytic systems. In order to focus on the key aspects and keep the presentation simple, we restrict ourselves in this paper to the discrete-time single-input systems.

Whereas the number of publications on the topic of static state feedback linearization is huge, the situation is different for the discrete-time case, see [5–11]. Except [11], all papers focus on smooth feedback.

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2. The algebra of functions

Consider a discrete-time nonlinear control system of the form

$$\sigma(x) = f(x, u), \tag{1}$$

where by $\sigma(x)$ is denoted the forward shift of x, alternatively written as x^+ and understood as $x(k+1)$, $f : X \times U \mapsto X$, the variables x and u are the coordinates of the state space $X \subseteq \mathbb{R}^n$ and the input space $U \subseteq \mathbb{R}$, respectively.

In the mathematical technique, called the algebra of functions, f in (1) is allowed to be non-smooth. The main elements of the technique, defined on the set S_X of vector functions with the domain being the state space X, are [1]:

1. relation of partial preorder, denoted by $\leq$, and equivalence, denoted by $\cong$,
2. binary operations, denoted by $\times$ and $\oplus$,
3. (non-symmetric) binary relation, denoted by Δ,
4. operators $\mathbf{m}$ and $\mathbf{M}$.

Definition 1 (*Relation of Partial Preorder*)**.** Given $\alpha, \beta \in S_X$, one says that $\alpha \leq \beta$ iff there exists a function γ such that $\beta(s) = \gamma(\alpha(s))$ for $\forall x \in X$.

If $\alpha \not\leq \beta$ and $\beta \not\leq \alpha$, then α and β are said to be incomparable.

Definition 2 (*Equivalence*)**.** If $\alpha \leq \beta$ and $\beta \leq \alpha$, then α and β are called equivalent, denoted by $\alpha \cong \beta$.

Besides the relations of partial preorder and equivalence, we use the generic notions, corresponding to the situation when the relation may be violated on a set of measure zero.

Note that the relation $\cong$ is reflexive, symmetric and transitive. The equivalence relation divides the set S_X into the *equivalence classes* containing the equivalent functions. If $S_X \backslash \cong$ is the set of all these equivalence classes, then the relation $\leq$ is *partial order* on this set. Recall that a lattice is a set with a partial order where every two elements α and β have a unique supremum (least upper bound) $\sup(\alpha, \beta)$ and an infimum (greatest lower bound) $\inf(\alpha, \beta)$. The equivalent definition of the lattice as an algebraic structure with two binary operations $\times$ and $\oplus$ may be given if for every two elements both operations are commutative and associative and moreover, $\alpha \times (\alpha \oplus \beta) = \alpha$, $\alpha \oplus (\alpha \times \beta) = \alpha$. The equivalence follows from the definition of the binary operations $\times$ and $\oplus$ as

$$\alpha \times \beta = \inf(\alpha, \beta), \qquad \alpha \oplus \beta = \sup(\alpha, \beta). \tag{2}$$

Therefore, the triple $(S_X \backslash \cong, \times, \oplus)$ is a lattice. In lattice theory it is customary not to operate with $\inf(\alpha, \beta)$ and $\sup(\alpha, \beta)$ but with binary operations $\times$ and $\oplus$, respectively.

In the simple cases the definition may be used to compute $\alpha \oplus \beta$. For the general case, see [1,12]. The rule for operation $\times$ is simple $(\alpha \times \beta)(x) = [\alpha^T(x), \beta^T(x)]^T$. However, the product may contain redundant (functionally dependent) components that have to be found and removed. Moreover, to simplify the computations, one is advised to replace the remaining components by equivalent but more simple functions, see more in [12].

Definition 3. (Binary relation Δ) Given $\alpha, \beta \in S_X$, α and β are said to form an (ordered) pair, denoted as $(\alpha, \beta) \in \Delta$, if there exists a function f_* such that

$$\beta(f(x, u)) = f_*(\alpha(x), u) \tag{3}$$

for all $(x, u) \in X \times U$.

The binary relation Δ may be given the following interpretation. One may ask what is the *necessary* information about $x(t)$ to compute $\beta(x(t+1))$ for arbitrary but known $u(t)$? The amount of the necessary information is displayed in function $\alpha(x)$, forming a pair with the function $\beta(x)$.

Obviously, given $\beta(x)$, there exist many functions $\alpha(x)$, forming a pair with $\beta(x)$, i.e. $(\alpha, \beta) \in \Delta$. The most important among them is the maximal function with respect to the relation $\leq$, denoted by $\mathbf{M}(\beta)$. In a similar manner, for given $\alpha(x)$, there exist many functions $\beta(x)$, forming a pair with $\alpha(x)$, i.e. $(\alpha, \beta) \in \Delta$. We will denote by $\mathbf{m}(\alpha)$ the minimal function among those functions (with respect to relation $\leq$). This yields the following definitions.

Definition 4. Operator $\mathbf{m}$, applied to a function α, is a function $\mathbf{m}(\alpha) \in S_X$ that satisfies the following two conditions

(i) $(\alpha, \mathbf{m}(\alpha)) \in \Delta$
(ii) if $(\alpha, \beta) \in \Delta$, then $\mathbf{m}(\alpha) \leq \beta$.

Definition 5. Operator $\mathbf{M}$, applied to a function β, is a function $\mathbf{M}(\beta) \in S_X$ that satisfies the following conditions

(i) $(\mathbf{M}(\beta), \beta) \in \Delta$
(ii) if $(\alpha, \beta) \in \Delta$, then $\alpha \leq \mathbf{M}(\beta)$.

Computation of $\mathbf{m}(\alpha)$. It has been proven that the function γ exists that satisfies the condition $(\alpha \times u) \oplus f \cong \gamma(f)$; define $\mathbf{m}(\alpha) \cong \gamma$, see [13]. Because the composition $\gamma(f)$ may be written as γ^+ and $\mathbf{m}(\alpha) \cong \gamma$, one may alternatively write the rule for computation of the operator $\mathbf{m}$ using a backward shift as follows:

$$\mathbf{m}(\alpha) \cong ((\alpha \times u) \oplus f)^-. \tag{4}$$

Computation of $\mathbf{M}(\beta)$. In the special case when the composite function $\beta(f(x, u))$ can be represented in the form

$$\beta(f(x, u)) = \sum_{i=1}^{d} a_i(x) b_i(u) \tag{5}$$

where $a_1(x), a_2(x), \ldots, a_d(x)$ are arbitrary functions and $b_1(u), b_2(u), \ldots, b_d(u)$ are linearly independent over $\mathbb{R}$, then

$$\mathbf{M}(\beta) := a_1 \times a_2 \times \cdots \times a_d. \tag{6}$$

For the general case, see [1].

Below we present two propositions that involve the coordinate transformation $\varphi : X \to Z$. Note that the transformation φ itself as well as its inverse φ^{-1} are defined generically, i.e. everywhere except perhaps on the set of zero measure. Moreover, note that neither φ nor φ^{-1} are necessarily smooth; see also Example 29 below.

Proposition 6 ([1] (pp. 81–82))**.** *Let α be a vector function on S_X and $\varphi : X \to Z$ a coordinate transformation. Then $\mathbf{m}(\alpha(\varphi)) \cong (\mathbf{m}(\alpha))(\varphi)$, $\mathbf{M}(\alpha(\varphi)) \cong (\mathbf{M}(\alpha))(\varphi)$.*

Proposition 7. *Let α and β be two vector functions on S_X and $\varphi : X \to Z$ a coordinate transformation. Then $(\alpha \oplus \beta)(\varphi) \cong \alpha(\varphi) \oplus \beta(\varphi)$.*

Proof. Note that by definition of operation $\oplus$ inequalities $(\alpha \oplus \beta)(\varphi) \geq \alpha(\varphi)$ and $(\alpha \oplus \beta)(\varphi) \geq \beta(\varphi)$ imply

$$(\alpha \oplus \beta)(\varphi) \geq \alpha(\varphi) \oplus \beta(\varphi) \tag{7}$$

and, furthermore, taking the composition with the function φ^{-1},

$$((\alpha \oplus \beta)(\varphi))(\varphi^{-1}) = \alpha \oplus \beta \geq (\alpha(\varphi) \oplus \beta(\varphi))(\varphi^{-1}). \tag{8}$$

Applying again (7) for the right-hand side of the above inequality, one gets

$$(\alpha(\varphi) \oplus \beta(\varphi))(\varphi^{-1}) \geq (\alpha(\varphi))(\varphi^{-1}) \oplus (\beta(\varphi))(\varphi^{-1}) = \alpha \oplus \beta. \tag{9}$$

Finally, (8) and (9) yield $(\alpha(\varphi) \oplus \beta(\varphi))(\varphi^{-1}) \cong \alpha \oplus \beta$, or $(\alpha \oplus \beta)(\varphi) \cong \alpha(\varphi) \oplus \beta(\varphi)$. □