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Controllability of driftless nonlinear time-delay systems

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ABSTRACT

The controllability of a class of nonlinear driftless time-delay systems is fully characterized for the first time. This result is obtained within a newly introduced geometric approach. Moreover, all those possible autonomous (or non controllable) elements, which can depend on the delayed variables, are also characterized when the system is not controllable and in consequence, a canonical form of those systems is derived.

1. Introduction

The aim of the present paper is to characterize completely the controllability properties of nonlinear driftless multi-input systems, with possible constant commensurate delays on the state variables within a new framework recently introduced in the literature in [1]. Easy checkable necessary and sufficient conditions are given to test the controllability of this class of delay systems. While the weaker property of accessibility of nonlinear time-delay systems was considered for the first time in [2] where a suitable definition of accessibility was proposed and a sufficient condition to test whether or not a given system is accessible was derived, the decomposition of a given system into an accessible part and an autonomous one has not been fully addressed (see also [3,4]). In the present paper a bicausal change of coordinates which allows to define such a decomposition is determined. This is done by further developing the geometric framework introduced in [1] and used in [5], by showing that the integrability of the submodule related to the accessibility property can be characterized by referring to an appropriate finite dimensional system associated to the given time-delay system instead of making use of Taylor series expansions as done in [1]. Furthermore we will refer to the stronger property of controllability since for driftless systems accessibility also implies controllability. Some unexpected situations are displayed as well: for instance, in opposition to the delay-free case [6–8], driftless nonlinear time-delay systems influenced by one single control variable, may be controllable even if the instantaneous state $x(t)$ has dimension $n \geq 2$.

The paper is organized as follows. In Section 2, some fundamental notions on time-delay systems are given as well as the definition of accessibility which were introduced in [1,2,9]. In Section 3, the controllability properties of the class of nonlinear driftless time-delay systems are discussed. An alternative easy necessary and sufficient condition for testing the controllability for this class of systems is proposed. Moreover, when the original system is not controllable, we show how to characterize all the autonomous functions associated to the system (which may depend on the delayed variables). Based on these autonomous functions, a standard decomposition into an autonomous subsystem and a controllable subsystem is then deduced. Examples show the technical details. Some preliminary results concerning the single input case were presented in [5].

2. Preliminaries and notations

In this paper, we characterize the controllability property of driftless time-delay systems of the form

$$\dot{x}(t) = \sum_{j=1}^{m} G_j(x(t), x(t-D), \ldots, x(t-sD))u_j(t) \tag{1}$$

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with D a constant commensurate delay, $s \geq 0$ an integer and the m functions $G_j(x(t), x(t-D), \dots, x(t-sD))$, $j \in [1, m]$, analytic in their arguments which satisfy $\text{rank}(G_1, \dots, G_m) = m$.[1] Denoting for notational compactness by $\mathbf{x}_{[s]}^T = (x^T(t), \dots, x^T(t-sD)) \in \mathbb{R}^{(s+1)n}$, the first $(s+1)n$ components of the state of the infinite dimensional system (1) with $\mathbf{x}_{[0]} = [x_{1,[0]}, \dots, x_{n,[0]}]^T = x(t) \in \mathbb{R}^n$, $\mathbf{u}_{[0]} = [u_{1,[0]}, \dots, u_{m,[0]}]^T = u(t) \in \mathbb{R}^m$, the instantaneous values of the state and input variables, (1) can be rewritten as

$$\Sigma : \dot{\mathbf{x}}_{[0]} = \sum_{j=1}^{m} G_j(\mathbf{x}_{[s]}) u_{j,[0]}. \tag{2}$$

Throughout the paper we will denote by $\mathbf{x}_{[s]}^T(-p) = (x^T(t-pD), \dots, x^T(t-sD-pD))$, while $\mathbf{u}_{[s]}$, $\mathbf{u}_{[s]}(-p)$, $\mathbf{z}_{[s]}$, and $\mathbf{z}_{[s]}(-p)$ are defined in a similar vein. Accordingly $x_{j,[0]}(-p) := x_j(t-pD)$, and $u_{i,[0]}(-p) := u_i(t-pD)$ denote respectively the j-th and i-th components of the instantaneous values of the state and input variables delayed by $\tau = pD$. When no confusion is possible the subindex will be omitted so that $\mathbf{x}$ will stand for $\mathbf{x}_{[s]}$ and $\mathbf{x}(-p)$ will stand for $\mathbf{x}_{[s]}(-p)$. Finally $\mathbf{u}^{[j]} := (\mathbf{u}^T, \dot{\mathbf{u}}^T, \dots, (\mathbf{u}^{(j)})^T)^T$ where by convention $\mathbf{u}^{[-1]} = \emptyset$.

As it is well known, when $n \geq 2$, single input delay-free systems (that is $m = 1$, $s = 0$ in Eq. (2) are never controllable. However if $s \geq 1$, system (2)) becomes infinite dimensional and as already noted in [1] and further discussed in [5] this is not true anymore for systems subject to delays. We will show how in the geometric framework introduced in [1] the study of the controllability properties of this class of system becomes very simple.

The following notation taken from [1,11,9,12] will be used:

- $\mathcal{K}$ denotes the field of meromorphic functions of a *finite* number of variables in $\{\mathbf{x}_{[0]}(-i), \mathbf{u}_{[0]}(-i), \dot{\mathbf{u}}_{[0]}(-i), \dots, \mathbf{u}_{[0]}^{(k)}(-i), i, k \in \mathbb{N}\}$;
- Given a function $f(\mathbf{x}_{[s]}, \mathbf{u}_{[j]}^{[k]})$, we will denote by $f(-l) = f(\mathbf{x}_{[s]}(-l), \mathbf{u}_{[j]}^{[k]}(-l))$;
- d is the standard differential operator;
- δ represents the backward time-shift operator: for $a(\mathbf{x}, \mathbf{u}^{[i]})$, $f(\mathbf{x}, \mathbf{u}^{[j]}) \in \mathcal{K}$:
$$\delta[a(\mathbf{x}, \mathbf{u}^{[i]})\, df(\mathbf{x}, \mathbf{u}^{[j]})] = a(\mathbf{x}(-1), \mathbf{u}^{[i]}(-1))\delta df(\mathbf{x}, \mathbf{u}^{[j]}) = a(\mathbf{x}(-1), \mathbf{u}^{[i]}(-1)) df(\mathbf{x}(-1), \mathbf{u}^{[j]}(-1));$$
- $\mathcal{K}(\delta]$ is the (left) ring of polynomials in δ with coefficients in $\mathcal{K}$. Every element of $\mathcal{K}(\delta]$ may be written as $\alpha(\delta] = \sum_{j=0}^{r_\alpha} \alpha_j(\mathbf{x}, \mathbf{u}^{[l_\alpha]})\delta^j$, with $\alpha_j(\cdot) \in \mathcal{K}$ and $r_\alpha = \deg(\alpha(\delta])$ the polynomial degree in δ. By convention $\alpha_i(\cdot) = 0$ for $i > r_\alpha$. Let $\beta(\mathbf{x}, \mathbf{u}^{[l_\beta]}) = \sum_{j=0}^{r_\beta} \beta_j(\mathbf{x}, \mathbf{u}^{[l_\beta]})\delta^j$ be an element of $\mathcal{K}(\delta]$ of polynomial degree r_β and set again $\beta_j(\cdot) = 0$ for $j > r_\beta$. Then addition and multiplication on this ring are defined by [9]
$$\alpha(\delta] + \beta(\delta] = \sum_{i=0}^{\max\{r_\alpha, r_\beta\}} (\alpha_i(\mathbf{x}, \mathbf{u}^{[l_\alpha]}) + \beta_i(\mathbf{x}, \mathbf{u}^{[l_\beta]}))\delta^i$$
$$\alpha(\delta]\beta(\delta] = \sum_{i=0}^{r_\alpha} \sum_{j=0}^{r_\beta} \alpha_i(\mathbf{x}, \mathbf{u}^{[l_\alpha]})\, \beta_j(\mathbf{x}(-i), \mathbf{u}^{[l_\beta]}(-i))\delta^{i+j}.$$
- Let for $i \in [1, j]$, $\tau_i(\mathbf{x}_{[l]})$ be vector fields defined in an open set $\Omega_l \subseteq \mathbb{R}^{n(l+1)}$. Then $\Delta = \text{span}\{\tau_i(\mathbf{x}_{[l]}), i = 1, \dots, j\}$ represents the distribution generated by the vector fields $\tau_i(\cdot)$ and defined in $\mathbb{R}^{n(l+1)}$. $\bar{\Delta}$ represents its involutive closure, that is, for any two vector fields $\tau_i(\cdot), \tau_j(\cdot) \in \bar{\Delta}$ then also the Lie bracket $[\tau_i, \tau_j] = \frac{\partial \tau_i}{\partial \mathbf{x}_{[l]}}\tau_j - \frac{\partial \tau_j}{\partial \mathbf{x}_{[l]}}\tau_i \in \bar{\Delta}$. Instead let for $i \in [1, j]$, $\tau_i(\mathbf{x}_{[l]}, \delta) \in \mathcal{K}^n(\delta]$, then $\Delta(\delta] = \text{span}_{\mathcal{K}(\delta]}\{\tau_i(\mathbf{x}, \delta), i = 1, \dots, j\}$ represents the module spanned by the generators $\tau_i(\mathbf{x}, \delta)$ over $\mathcal{K}(\delta]$, that is any $\tau(\mathbf{x}, \delta) \in \Delta(\delta]$ can be expressed as $\tau(\mathbf{x}, \delta) = \sum_{i=1}^{j} \tau_i(\mathbf{x}, \delta)\alpha_i(\mathbf{x}, \delta)$.

To the dynamics (2) we can associate its differential form representation defined as

$$\Sigma_L : d\dot{\mathbf{x}}_{[0]} = f(\mathbf{x}_{[s]}, \mathbf{u}_{[0]}, \delta)\, d\mathbf{x}_{[0]} + g_1(\mathbf{x}_{[s]}, \delta)\, d\mathbf{u}_{[0]} \tag{3}$$

with

$$f(\mathbf{x}_{[s]}, \mathbf{u}_{[0]}, \delta) = \sum_{i=0}^{s} \sum_{j=1}^{m} u_{j,[0]} \frac{\partial G_j(\mathbf{x}_{[s]})}{\partial \mathbf{x}_{[0]}(-i)} \delta^i, \tag{4}$$

$$\begin{aligned} g_1(\mathbf{x}_{[s]}, \delta) &= (g_{11}(\mathbf{x}_{[s]}, \delta), \dots, g_{1m}(\mathbf{x}_{[s]}, \delta)) \\ &= \left(G_1(\mathbf{x}_{[s]}), \dots G_m(\mathbf{x}_{[s]})\right) = g_1^0(\mathbf{x}). \end{aligned} \tag{5}$$

Iteratively let us consider [2]

$$\begin{aligned} g_{i+1,j}(\mathbf{x}, \mathbf{u}^{[i-1]}, \delta) &= f(\mathbf{x}, \mathbf{u}, \delta) g_{i,j}(\mathbf{x}, \mathbf{u}^{[i-2]}, \delta) \\ &\quad - \dot{g}_{i,j}(\mathbf{x}, \mathbf{u}^{[i-2]}, \delta). \end{aligned} \tag{6}$$

As underlined in [2], the $g_{i+1,j}(\mathbf{x}, \mathbf{u}^{[i-1]}, \delta)$'s are strictly related to the accessibility properties of a system with delays. Accordingly we can introduce the following accessibility submodules

Definition 1. The accessibility submodules $\mathcal{R}_i$ of Σ, are defined as

$$\mathcal{R}_i(\mathbf{x}, \mathbf{u}^{[i-2]}, \delta) = \text{span}_{\mathcal{K}(\delta]}\{g_1(\mathbf{x}, \delta) \dots g_i(\mathbf{x}, \mathbf{u}^{[i-2]}, \delta)\}, \quad i \geq 1.$$

The following result generalizes to the multi-input case the one given in [1].

Proposition 1. *If $g_{i+1,j}(\cdot) \in \mathcal{R}_i$, then we have $g_{i+k+1,j}(\cdot) \in \mathcal{R}_{i+k}$, for $\forall k \geq 0$. Furthermore if $g_{i+1,j}(\cdot) \in \mathcal{R}_i, \forall j \in [1, m]$, then $g_{i+k,j}(\cdot) \in \mathcal{R}_i$, for $\forall k \geq 0$, and $\forall j \in [1, m]$.*

In such a framework in [1,13] the definitions of Extended Lie bracket and Extended Lie derivative were introduced for dealing with time delay systems. These definitions which differ from the delayed state bracket introduced in [14] and delayed Lie derivative in [15,16], will be used in the present context with reference to the controllability property.

Let $r(\mathbf{x}, \delta) = \sum_{j=0}^{\bar{s}} r^j(\mathbf{x})\delta^j$, and set $r^{\bar{s}+j}(\mathbf{x}) = 0$ for any $j > 0$.

Definition 2. Given the function $\tau(\mathbf{x}_{[s]})$ and the element $r(\mathbf{x}, \delta) = \sum_{j=0}^{\bar{s}} r^j(\mathbf{x})\delta^j$, the Extended Lie derivative $L_{r^j(\mathbf{x})}\tau(\mathbf{x}_{[s]})$ is defined as

$$L_{r^j(\mathbf{x})}\tau(\mathbf{x}_{[s]}) = \sum_{l=0}^{j} \frac{\partial \tau(\mathbf{x}_{[s]})}{\partial \mathbf{x}_{[0]}(-l)} r^{j-l}(\mathbf{x}(-l)). \tag{7}$$

Definition 3. Let $r_i(\mathbf{x}, \delta) = \sum_{j=0}^{\bar{s}} r_i^j(\mathbf{x})\delta^j$, $i = 1, 2$. For any $k, l \geq 0$, the Extended Lie bracket $[r_1^k(\cdot), r_2^l(\cdot)]_{E_i}$, on $\mathbb{R}^{(i+1)n}$, is defined as

$$[r_1^k(\cdot), r_2^l(\cdot)]_{E_0} = \left(L_{r_1^k(\mathbf{x})} r_2^l(\mathbf{x}) - L_{r_2^l(\mathbf{x})} r_1^k(\mathbf{x})\right)^T \frac{\partial}{\partial \mathbf{x}_{[0]}}. \tag{8}$$

$$[r_1^k(\cdot), r_2^l(\cdot)]_{E_i} = \sum_{j=0}^{\min(k,l,i)} \left([r_1^{k-j}(\cdot), r_2^{l-j}(\cdot)]_{E_0}\right)^T_{(\mathbf{x}(-j))} \frac{\partial}{\partial \mathbf{x}_{[0]}(-j)}. \tag{9}$$

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[1] Let us recall that any system of the form

$$\dot{x}(t) = \sum_{j=1}^{m} G_j(x(t), x(t-\tau_1), \dots, x(t-\tau_p)) u_j(t)$$

where $\tau_i, i \in [1, p]$ are constant commensurate delays can be rewritten in the form (1). In fact, as well known [10], the delays being commensurate, they can be expressed as integer multiples of a common constant commensurate delay D that is $\tau_i = l_i \cdot D$, $i \in [1, p]$. Denoting by $s = \max\{l_1, \dots, l_p\}$ one recovers (1).