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Risk sensitive and LEG filtering problems are not equivalent

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ABSTRACT

Filtering problems with general exponential quadratic criteria are investigated for Gauss–Markov processes. In this setting, the linear exponential Gaussian and risk sensitive filtering problems are solved and it is shown that they may have different solutions.

1. Introduction

The so-called linear exponential Gaussian (LEG) and risk sensitive (RS) filtering problems involve criteria which are exponentials of integral cost functionals. Before our paper [1], numerous results had already been reported in specific models, specially around Markov models, but without exhibiting the relationship between these two problems. See, e.g., Whittle [2], Speyer et al. [3], Elliott et al. [4–6] for contributions. In our paper [1], we have solved the LEG and RS filtering problems for general Gaussian processes in the particular setting where the functional in the exponential is a *singular* quadratic functional. Moreover we have proved that actually in this case the solutions coincide. In the present paper the problems are revisited for Gauss–Markov processes but with a *non-singular* quadratic functional in the exponential. In this setting the solutions are exhibited and we propose an example to show that they may be different.

It what follows all random variables and processes are defined on a given stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ satisfying the usual conditions and processes are $(\mathcal{F}_t)$-adapted. We deal with a signal process $X = (X_t,\ t \geq 0)$ in $\mathbb{R}$ governed by the linear equation

$$\mathrm{d}X_t = a_t X_t \mathrm{d}t + \mathrm{d}B_t, \quad X_0 = 0, \tag{1}$$

and an observation process $Y = (Y_t,\ t \geq 0)$ in $\mathbb{R}$ governed by the linear equation

$$\mathrm{d}Y_t = A_t X_t \mathrm{d}t + \mathrm{d}\widetilde{B}_t, \quad Y_0 = 0,\ t \geq 0. \tag{2}$$

Here $a = (a_t,\ t \geq 0)$ and $A = (A_t,\ t \geq 0)$ are continuous real-valued deterministic functions, $B = (B_t,\ t \geq 0)$ and $\widetilde{B} = (\widetilde{B}_t,\ t \geq 0)$ are independent 1D standard Brownian motions. Clearly the pair (X, Y) is Gaussian.

For a given continuous deterministic function $\Lambda = (\Lambda_s,\ 0 \leq s \leq T)$ with values in the set of nonnegative definite symmetric 2×2 matrices

$$\Lambda_s = \begin{pmatrix} \Lambda_{11}(s) & \Lambda_{12}(s) \\ \Lambda_{12}(s) & \Lambda_{22}(s) \end{pmatrix},$$

such that $\Lambda_{22}(s) \neq 0$, let us denote by $\bar{h} \in \mathcal{H}$ the solution of the *LEG type filtering problem*:

$$\bar{h} = \underset{h\in\mathcal{H}}{\operatorname{argmin}} \frac{1}{\mu} \ln \mathbb{E}\left[\exp\left\{\frac{\mu}{2}\int_0^T (X_s h_s)\Lambda_s \begin{pmatrix} X_s \\ h_s \end{pmatrix} \mathrm{d}s\right\}\right]. \tag{3}$$

In this definition μ is a real parameter and $h = (h_s,\ 0 \leq s \leq T) \in \mathcal{H}$ means that h is a $(\mathcal{Y}_s)$-adapted continuous process where $(\mathcal{Y}_s)$ is the natural filtration of Y, *i.e.*, $\mathcal{Y}_s = \sigma(\{Y_u,\ 0 \leq u \leq s\})$, $0 \leq s \leq T$.

We can also define $\hat{h}$ as a solution of the following recursive equation, which is the basic definition of the *RS type filtering problem*:

$$\begin{aligned} \widehat{h}_t = &\underset{g\in\mathcal{Y}_t}{\operatorname{argmin}} \frac{1}{\mu} \ln \mathbb{E}\left[\exp\left\{\frac{\mu}{2}(X_t g)\Lambda_t \begin{pmatrix} X_t \\ g \end{pmatrix}\right.\right. \\ &\left.\left. + \frac{\mu}{2}\int_0^t (X_s \widehat{h}_s)\Lambda_s \begin{pmatrix} X_s \\ \widehat{h}_s \end{pmatrix} \mathrm{d}s\right\} \Big/ \mathcal{Y}_t\right], \end{aligned} \tag{4}$$

where $g \in \mathcal{Y}_t$ means that g is a $\mathcal{Y}_t$-measurable variable.

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It is clear that *risk-neutral* versions of these two problems (namely, dropping the exponentials in definitions (3)–(4), *i.e.*, simply with quadratic criteria) are "equivalent". Actually both the solutions coincide with a weighted Kalman filter:

$$\bar{h}_t = \widehat{h}_t = -\frac{\Lambda_{12}(t)}{\Lambda_{22}(t)} \cdot \pi_t(X),$$

where for any process $\eta = (\eta_t,\ t \in [0, T])$ such that $\mathbb{E}|\eta_t| < +\infty$, the notation $\pi_t(\eta)$ is used for the conditional expectation of η_t given the σ-field $\mathcal{Y}_t$,

$$\pi_t(\eta) = \mathbb{E}(\eta_t/\mathcal{Y}_t).$$

One question that we want to discuss in this paper is the possible "equivalence" of problems (3) and (4). In our paper [1], we have considered the case when the quadratic functional involved in the exponential is *singular*, namely when matrices Λ_s are singular, *i.e.*, $\Lambda_{11} = \Lambda_{22} = -\Lambda_{12}$, and hence $(X_s\, h_s)\Lambda_s \binom{X_s}{h_s} = \Lambda_{11}(s)(X_s - h_s)^2$, which means that only squares of filtering errors are taken into account in the criteria; we have shown that in this setup the equality $\bar{h} = \hat{h}$ holds, even in a non-Markovian setting. Here a simple example where $\bar{h} \neq \widehat{h}$ is proposed below which shows that if the quadratic functional is *nonsingular* then the answer may be negative even for the Markovian model (1)–(2). Notice that such a setup includes criteria which take into account possible estimating costs evaluated as squares h_s^2 of the filters.

The paper is organized as follows. Section 2 contains the statements providing the solutions of LEG and RS filtering problems in the nonsingular setting. In Section 3 the announced example which shows the discrepancy between the two filtering problems is analyzed. A comparison of the different filters including a discussion about their robustness is also proposed. Then preparing for the analysis of the filtering problems, in Section 4 a Cameron–Martin type formula for the *conditional Laplace transform* of a quadratic functional of the involved signal process is derived. Finally Section 5 contains the proofs of the announced results.

2. Solution of the filtering problems with exponentials of integral functional criteria

Here we formulate the statements providing the solutions of the LEG and RS filtering problems. Their proofs are given in Section 5.

2.1. Solution of the LEG filtering problem

Let us formulate the following condition (C_μ^*):

(C_μ^*) the forward and backward Riccati equations:

$$\dot{\bar{\gamma}}_{xx} = 2a_t\bar{\gamma}_{xx} + 1 - \bar{\gamma}_{xx}^2[A_t^2 - \mu\Lambda_{11}], \quad \bar{\gamma}_{xx}(0) = 0, \tag{5}$$

$$\dot{\Gamma} = -\frac{det(\Lambda)}{\Lambda_{22}} - 2\left(a_t + \mu\bar{\gamma}_{xx}\frac{det(\Lambda)}{\Lambda_{22}}\right)\Gamma - \mu\Gamma^2\bar{\gamma}_{xx}^2\left[A_t^2 - \mu\frac{\Lambda_{12}^2}{\Lambda_{22}}\right], \quad \Gamma(T,T) = 0, \tag{6}$$

have unique nonnegative and bounded solutions $(\bar{\gamma}_{xx}(t),\ 0 \le t \le T)$ and $(\Gamma(T,t),\ 0 \le t \le T)$.

Notice that for all μ *negative* condition (C_μ^*) is satisfied and if μ is *positive*, it is satisfied for μ sufficiently small, for example, those such that for any $t \le T$ $A_t^2 - \mu\Lambda_{11}(t)$ is nonnegative (*cf.* Lemma 2 [1]).

Proposition 1. *Suppose that condition (C_μ^*) is satisfied. Let $\bar{h} = (\bar{h}_t,\ 0 \le t \le T)$ such that:*

$$\bar{h}_t = -\frac{\Lambda_{12}(t)}{\Lambda_{22}(t)}(1 - \mu\bar{\gamma}_{xx}(t)\Gamma(T,t))Z_t^{\bar{h}}, \tag{7}$$

where $\bar{\gamma}_{xx}$ and $\Gamma(T,\cdot)$ are the solutions of the Riccati equations (5) and (6) respectively and $Z^{\bar{h}} = (Z_t^{\bar{h}},\ 0 \le t \le T)$ is the solution of the following equation:

$$\mathrm{d}Z_t^{\bar{h}} = \left[a + \mu\frac{\bar{\gamma}_{xx}}{\Lambda_{22}}(det(\Lambda) + \mu\Lambda_{12}^2\bar{\gamma}_{xx}\Gamma)\right]Z_t^{\bar{h}}\mathrm{d}t + A\bar{\gamma}_{xx}[\mathrm{d}Y_t - AZ_t^{\bar{h}}\mathrm{d}t]. \tag{8}$$

Then $\bar{h}$ is the solution of the LEG filtering problem (3) and moreover, the corresponding optimal risk is given by

$$\frac{1}{\mu}\ln\mathbb{E}\left[\exp\left\{\frac{\mu}{2}\int_0^T (X_s\bar{h}_s)\Lambda_s\begin{pmatrix}X_s\\ \bar{h}_s\end{pmatrix}\mathrm{d}s\right\}\right] = \frac{1}{2}\int_0^T \bar{\gamma}_{xx}(s)\Lambda_{11}(s)\mathrm{d}s + \frac{1}{2}\int_0^T \Gamma(T,s)A_s^2\bar{\gamma}_{xx}^2(s)\mathrm{d}s.$$

Remark 1. (i) It is clear that in the singular case where $\Lambda_{11} = \Lambda_{22} = -\Lambda_{12}$, Eq. (6) implies that $\Gamma \equiv 0$ and therefore $Z^{\bar{h}} = \bar{h}$ (*cf.* [1]).
(ii) But in the general case Γ may depend on T and as a consequence, $\bar{h}_t$ may also depend on T. An example of such a dependence will be given below. Of course, by its definition $\widehat{h}_t$ does not depend on T and hence $\bar{h} \neq \widehat{h}$ in this example.

2.2. Solution of the RS filtering problem

Let us formulate the following condition (C_μ^{**}):

(C_μ^{**}) the Riccati equation (5) has a unique, nonnegative and bounded solution on $[0, T]$ such that for $0 \le t \le T$

$$1 - \mu\bar{\gamma}_{xx}(t)\Lambda_{11}(t) > 0.$$

Proposition 2. *Suppose that condition (C_μ^{**}) is satisfied. Let $\widehat{h} = (\widehat{h}_t,\ 0 \le t \le T)$ such that:*

$$\widehat{h}_t = -\Lambda_{12}(t)[\Lambda_{22}(t) - \mu\bar{\gamma}_{xx}(t)det(\Lambda_t)]^{-1}Z_t^{\widehat{h}}, \tag{9}$$

where $Z_t^{\widehat{h}} = (Z_t^{\widehat{h}},\ 0 \le t \le T)$ is the solution of the following equation:

$$\mathrm{d}Z_t^{\widehat{h}} = \left[a_t + \mu\bar{\gamma}_{xx}det(\Lambda)\frac{1 - \mu\bar{\gamma}_{xx}\Lambda_{11}det(\Lambda)}{\Lambda_{22} - \mu\bar{\gamma}_{xx}det(\Lambda)}\right]Z_t^{\widehat{h}}\mathrm{d}t + A_t\bar{\gamma}_{xx}[\mathrm{d}Y_t - A_tZ_t^{\widehat{h}}\mathrm{d}t]. \tag{10}$$

Then $\widehat{h}$ is the solution of the RS filtering problem (4).

Remark 2. (i) It is clear that for the singular case where $\Lambda_{11} = \Lambda_{22} = -\Lambda_{12}$, equalities (9) and (10) imply that $\widehat{h} = Z^{\widehat{h}} = Z^{\bar{h}} = \bar{h}$ (*cf.* [1]).
(ii) Let us emphasize that, of course, $\widehat{h}_t$ does not depend on T and so generally $\widehat{h}_t \neq \bar{h}_t$.

3. Comparison of the filters: an example

To illustrate the possible dependence of the solution of the LEG filtering problem on T and so the discrepancy between LEG and RS filtering problems we propose to take