Systems & Control Letters 58 (2009) 646–651

Contents lists available at ScienceDirect

Systems & Control Letters

journal homepage: www.elsevier.com/locate/sysconle

Irreducibility, reduction and transfer equivalence of nonlinear input–output equations on homogeneous time scales[☆]

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ARTICLE INFO

Article history:
Received 12 May 2008
Received in revised form
18 March 2009
Accepted 20 April 2009
Available online 20 May 2009

Keywords:
Nonlinear control systems
Continuous-time systems
Discrete-time systems
Reduction
Polynomials
Time scale
σ-differential field

ABSTRACT

The purpose of this paper is to present a necessary and sufficient condition for irreducibility of nonlinear input–output delta differential equations. The condition is presented in terms of the common left divisor of two differential polynomials describing the behaviour of the system defined on a homogenous time scale. The concept of reduction is explained. Subsequently, the definition of *transfer equivalence* based upon the notion of an irreducible differential form of the system is introduced, inspired by the analogous definition for continuous-time systems.

1. Introduction

In the realization problem of a higher order input–output (i/o) difference or differential equation one is looking for the state equations that would generate a given i/o equation. In [1] and [2] it was shown for discrete-time and continuous-time systems, respectively, that to obtain a minimal (i.e. both observable and accessible) realization of an i/o equation, the i/o equation has to be irreducible. So, the first task in solving the realization problem is to reduce the i/o equation when necessary. Irreducibility may be checked and the reduced system can be found in many different ways; see [3,4,1] for the discrete-time case and [2,5,6] for continuous-time case. One possibility is to associate with the control system two polynomials, defined over the difference (or differential, in the continuous time case) field of meromorphic functions, pretty much in a similar manner to how it has be done in the linear case. Then, in practical terms, checking irreducibility requires finding the left common divisor of these two polynomials; see [20]. In both discrete-time and continuous-time cases, the criteria for checking irreducibility are based on similar ideas. The main difference is that the multiplication rules between the shift (or differentiation in the continuous-time case) operator and an element of the difference (resp. differential) field are different, and they yield different noncommutative rings of polynomials. Therefore, it seems natural to try to unify the results for discrete-time and continuous-time cases into one result from which both would follow. However, in the discrete-time case our formalism yields a description based on the difference operator in opposition to the shift operator which was used in [3]. Recently, a delta-NARX model has been suggested for modelling the nonlinear control systems and it has been applied to the identification of a test problem (a van der Pol oscillator) [7]. Comparison was made with the standard shift operator based NARX model. It was demonstrated that a delta-NARX model improves the numerical properties of structure detection and provides a model that is closely linked to the continuous-time system in terms of both parameters and structure. These properties of delta-domain models were earlier shown to hold for linear models in [8].

The calculus on time scales seems to be a perfect language for a unification of continuous-time and discrete-time cases; see [9]. A time scale is a model of time. It is an arbitrary closed subset of the real line. Besides of standard cases of the whole line (continuous-time case) and the set of integers (discrete-time case), there are

☆ This paper was partially presented at the 7th IFAC Symposium on Nonlinear Control Systems [Z. Bartosiewicz, Ü. Kotta, E. Pawłuszewicz, M. Wyrwas, Irreducibility conditions for nonlinear input–output equations on homogeneous time scales, in: Preprints of 7th IFAC Symposium on Nonlinear Control Systems, 2007.].

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doi:10.1016/j.sysconle.2009.04.006

many examples of time models that may be partly continuous and partly discrete. One of the main concepts in the time scale analysis is the delta derivative, which is a generalization of the classical (time) derivative in the continuous time and the finite forward difference in the discrete time. Other approaches that allow the unification of continuous-time and discrete-time cases can be found in [10,11].

The purpose of this paper is to present a necessary and sufficient condition for irreducibility of nonlinear i/o delta differential equation on a homogeneous time scale that accommodates both the discrete-time and continuous-time cases as special cases. We will show that one can associate with such a system two polynomials over the σ-differential field, that belong to a noncommutative skew polynomial ring. If the system is reducible, the reduction procedure can be applied to transform the system into the irreducible form. Using time scale calculus we will unify the solution of the problem of reduction for given single-input single-output nonlinear control systems. Next, the definition of transfer equivalence for nonlinear systems defined on homogeneous time scales will be given. The notion of transfer equivalence plays a crucial role in the realization problem.

2. Calculus on a time scale

For a general introduction to calculus on time scales, see [9]. Here we give only those notions and facts that we need in our paper.

A *time scale* $\mathbb{T}$ is an arbitrary nonempty closed subset of the set $\mathbb{R}$ of real numbers. The standard cases comprise $\mathbb{T}=\mathbb{R}$, $\mathbb{T}=\mathbb{Z}$, $\mathbb{T}=h\mathbb{Z}$ for $h>0$, but also $\mathbb{T}=q^{\mathbb{Z}}:=\{q^k : k\in\mathbb{Z}\}\cup\{0\}$, $q>1$, is a time scale. The *forward jump operator* $\sigma:\mathbb{T}\to\mathbb{T}$ is defined by $\sigma(t)=\inf\{s\in\mathbb{T}: s>t\}$ for $t\in\mathbb{T}$, while the *backward jump operator* $\rho:\mathbb{T}\to\mathbb{T}$ is defined by $\rho(t)=\sup\{s\in\mathbb{T}: s<t\}$. Additionally, we set $\sigma(\max\mathbb{T})=\max\mathbb{T}$ if there exists a finite $\max\mathbb{T}$ and $\rho(\min\mathbb{T})=\min\mathbb{T}$ if there exists a finite $\min\mathbb{T}$.

The graininess function $\mu:\mathbb{T}\to[0,\infty)$ is defined by $\mu(t)=\sigma(t)-t$, for all $t\in\mathbb{T}$. If $\mu\equiv$ const then a time scale $\mathbb{T}$ is called *homogeneous*. From now on we assume that the time scale $\mathbb{T}$ is homogenous.

Definition 1. Let $f:\mathbb{T}\to\mathbb{R}$ be a function and $t\in\mathbb{T}$. Then the *delta derivative* of f at the point t is defined to be the number $f^{\Delta}(t)$ (provided it exists) with the property that for each $\epsilon>0$ there exists a neighbourhood $\mathcal{U}$ of t in $\mathbb{T}$ such that $\left|f(\sigma(t))-f(s)-f^{\Delta}(t)[\sigma(t)-s]\right|\leqslant\epsilon|\sigma(t)-s|$, for all $s\in\mathcal{U}$. Moreover, we say that f is *delta differentiable* (on $\mathbb{T}$) provided $f^{\Delta}(t)$ exists for all $t\in\mathbb{T}$.

Remark 2. A delta derivative is a natural extension of the time derivative in the continuous-time case and the forward difference operator in the discrete-time case.

Let $f:\mathbb{T}\to\mathbb{R}$. Define $f^{\sigma}:=f\circ\sigma$. Then we have $f^{\sigma}=f+\mu f^{\Delta}$.

Theorem 3 (*Chain Rule*). *Let $f:\mathbb{R}\to\mathbb{R}$ be continuously differentiable and suppose $g:\mathbb{T}\to\mathbb{R}$ is delta differentiable. Then $f\circ g:\mathbb{T}\to\mathbb{R}$ is delta differentiable and $(f\circ g)^{\Delta}(t)=\left\{\int_0^1 f'(g(t)+h\mu(t)g^{\Delta}(t))\mathrm{d}h\right\}g^{\Delta}(t)$.*

For a function $f:\mathbb{T}\to\mathbb{R}$ we define its second delta derivative $f^{[2]}:=f^{\Delta\Delta}$ provided that f^{Δ} is delta differentiable on $\mathbb{T}$. Similarly we define higher order delta derivatives $f^{[n]}$. Let us denote $f^{\Delta\sigma}:=\left(f^{\Delta}\right)^{\sigma}$ and $f^{\sigma\Delta}:=(f^{\sigma})^{\Delta}$.

Remark 4. Let f and f^{Δ} be delta differentiable functions. Then for a homogeneous time scale $\mathbb{T}$ we have $f^{\Delta\sigma}=f^{\sigma\Delta}$.

Let $\sigma^n:=\underbrace{\sigma\circ\cdots\circ\sigma}_{n-\text{times}}$ and $f^{\sigma^n}:=f\circ\sigma^n$. By the induction principle one can prove that if f is a delta differentiable function defined on a homogeneous time scale $\mathbb{T}$, then $f^{\sigma^n}=\sum_{k=0}^{n}\binom{n}{k}\mu^k f^{[k]}$.

3. σ-differential field $\mathcal{K}$ associated with the i/o delta differential equation

For $f:\mathbb{T}\to\mathbb{R}$ and $i\leqslant k$ let $f^{[i\ldots k]}:=\left(f^{[i]},\ldots,f^{[k]}\right)$. Let $y:\mathbb{T}\to\mathbb{R}$ and $u:\mathbb{T}\to\mathbb{R}$ be delta differentiable functions.

Consider a single-input single-output dynamical system described by a higher order input–output (i/o) delta-differential equation on the homogeneous time scale $\mathbb{T}$:

$$y^{[n]}=\phi\left(y^{[0\ldots n-1]},u^{[0\ldots s]}\right), \tag{1}$$

where u is a scalar input variable, $y\in\mathcal{Y}\subset\mathbb{R}$ is a scalar output variable, ϕ is a real meromorphic function, and n and s are nonnegative integers such that $s<n$. Let $\varphi(y^{[0\ldots n]},u^{[0\ldots s]}):=y^{[n]}-\phi\left(y^{[0\ldots n-1]},u^{[0\ldots s]}\right)$. Then Eq. (1) can be rewritten as follows: $\varphi\left(y^{[0\ldots n]},u^{[0\ldots s]}\right)=0$.

Assume that

$$1+\sum_{i=1}^{n}(-1)^{i+1}\mu^i\frac{\partial\phi}{\partial y^{[n-i]}}\not\equiv 0\quad\text{or}\quad\sum_{j=0}^{s}(-1)^j\mu^{j+2}\frac{\partial\phi}{\partial u^{[s-j]}}\not\equiv 0 \tag{2}$$

is satisfied generically, i.e. (2) holds almost everywhere except on a set of zero measure.

Let $\mathcal{R}$ denote the ring of analytic functions in a finite number of the variables $\{y^{[0\ldots n-1]},u^{[k]}:k\geqslant 0\}$. The operators $\sigma:\mathcal{R}\to\mathcal{R}$ and $\Delta:\mathcal{R}\to\mathcal{R}$ are defined as follows:

$$\sigma(F)\left(y^{[0\ldots n-1]},u^{[0\ldots k+1]}\right):=F\left(\left(y^{[0\ldots n-1]}\right)^{\sigma},\left(u^{[0\ldots k]}\right)^{\sigma}\right), \tag{3}$$

where $(y^{[0\ldots n-1]})^{\sigma}=(y+\mu y^{[1]},\ldots,y^{[n-1]}+\mu\phi(y^{[0\ldots n-1]},u^{[0\ldots s]}))$, $(u^{[0\ldots k]})^{\sigma}=u^{[0\ldots k]}+\mu u^{[1\ldots k+1]}$, for $k\geqslant 0$, and

$$\begin{aligned}\Delta(F)\left(y^{[0\ldots n-1]},u^{[0\ldots k+1]}\right):=\int_0^1\Big\{&\operatorname{grad}F\Big(y^{[0\ldots n-1]}\\&+h\mu\left[y^{[1\ldots n-1]},\phi\left(y^{[0\ldots n-1]},u^{[0\ldots s]}\right)\right],u^{[0\ldots k]}+h\mu u^{[1\ldots k+1]}\Big)\\&\times\left[y^{[1\ldots n-1]},\phi\left(y^{[0\ldots n-1]},u^{[0\ldots s]}\right),u^{[1\ldots k+1]}\right]^{\mathrm{T}}\Big\}\mathrm{d}h.\end{aligned} \tag{4}$$

We will use $\sigma(F)$ and F^{σ} to denote the action of σ on F. Similarly, both $\Delta(F)$ and F^{Δ} will be used interchangeably. Note that (2) is equivalent to the fact that the map $\left(y^{[0\ldots n-1]},u^{[0\ldots s]}\right)\mapsto\left(y^{[0\ldots n-1]},u^{[0\ldots s]}\right)+\mu\left(y^{[1\ldots n-1]},\phi(y^{[0\ldots n-1]},u^{[0\ldots s]}),u^{[1\ldots s+1]}\right)$ is a submersion (see [12]) and this implies that σ is a monomorphism of $\mathcal{R}$. Let $\mathcal{K}$ be a quotient field of the ring $\mathcal{R}$. The elements of $\mathcal{K}$ are meromorphic functions depending on a finite number of variables from $\{y^{[0\ldots n-1]},u^{[k]}:k\geqslant 0\}$. The operators σ and Δ can be extended to $\mathcal{K}$ using the same formulas (3) and (4). The operator Δ satisfies a generalization of the Leibniz rule:

$$(FG)^{\Delta}=F^{\sigma}G^{\Delta}+F^{\Delta}G, \tag{5}$$

for $F,G\in\mathcal{K}$. In noncommutative algebra the derivation satisfying rule (5) is called a "σ-*derivation*" (for example, see [13]). Therefore $\mathcal{K}$ is a field equipped with a σ-derivation Δ such that σ is a monomorphism of $\mathcal{K}$. The field $\mathcal{K}$ with σ-derivation Δ is a σ-*differential field*. Since σ is one-to-one, one can show that there is a σ-differential overfield $\mathcal{K}^*$ [14], called the *inversive closure* of $\mathcal{K}$, such that σ can be extended to $\mathcal{K}^*$, and this extension is an automorphism of $\mathcal{K}^*$ (see [13]). Therefore we assume that the inversive closure of σ-differential field $\mathcal{K}$ is given and we will use the same symbol to denote the σ-differential field and its inversive closure.