Available online at www.sciencedirect.com
ScienceDirect
Systems & Control Letters 57 (2008) 670–679
systems & control letters
www.elsevier.com/locate/sysconle

On system state equipartitioning and semistability in network dynamical systems with arbitrary time-delays[☆]

VijaySekhar Chellaboina[a,1], Wassim M. Haddad[b,*], Qing Hui[b,2], Jayanthy Ramakrishnan[a,3]

[a] *MABE Department, University of Tennessee, Knoxville, TN 37996-2210, United States*
[b] *School of Aerospace Engineering, Georgia Institute of Technology, Atlanta, GA 30332-0150, United States*

Received 7 July 2006; received in revised form 21 January 2008; accepted 27 January 2008
Available online 20 March 2008

Abstract

In many applications involving multiagent systems, groups of agents are required to agree on certain quantities of interest. In particular, it is important to develop consensus protocols for networks of dynamic agents with directed information flow, switching network topologies, and possible system time-delays. In this paper, we use compartmental dynamical system models to characterize dynamic algorithms for linear and nonlinear networks of dynamic agents in the presence of inter-agent communication delays that possess a continuum of *semistable* equilibria, that is, protocol algorithms that guarantee convergence to Lyapunov stable equilibria. In addition, we show that the steady-state distribution of the dynamic network is uniform, leading to system state equipartitioning or consensus. These results extend the results in the literature on consensus protocols for linear balanced networks to linear and nonlinear unbalanced networks with time-delays.

Keywords: Nonnegative systems; Compartmental models; Network systems; Networks with time-delays; Semistability; State equipartition; Consensus protocols

1. Introduction

Modern complex dynamical systems are highly interconnected and mutually interdependent, both physically and through a multitude of information and communication networks. By properly formulating these systems in terms of subsystem interaction involving energy/mass transfer, the dynamical models of many of these systems can be derived from mass, energy, and information balance considerations that involve dynamic states whose values are nonnegative. Hence, it follows from physical considerations that the state trajectory of such systems remains in the nonnegative orthant of the state space for nonnegative initial conditions. Such systems are commonly referred to as *nonnegative dynamical systems* in the literature [1,2]. A subclass of nonnegative dynamical systems are *compartmental systems* [2–5]. Compartmental systems involve dynamical models that are characterized by conservation laws (e.g., mass and energy) capturing the exchange of material between coupled macroscopic subsystems known as compartments. Each compartment is assumed to be kinetically homogeneous, that is, any material entering the compartment is instantaneously mixed with the material of the compartment. The range of applications of nonnegative systems and compartmental systems includes biological and physiological systems [5,6], chemical reaction systems [7,8], queuing systems [9], large-scale systems [10], stochastic systems (whose state variables represent probabilities) [9], ecological systems [11], economic systems [12], demographic systems [5], telecommunications systems [13], transportation systems, power systems, thermodynamic systems [14], and structural vibration systems, to cite but a few examples.

A key physical limitation of compartmental systems is that transfers between compartments are not instantaneous and realistic models for capturing the dynamics of such systems

[☆] This research was supported in part by the National Science Foundation under Grant ECS-0551947 and the Air Force Office of Scientific Research under Grant FA9550-06-1-0240.
* Corresponding author. Tel.: +1 404 894 1078; fax: +1 404 894 2760.
E-mail addresses: chellaboina@utk.edu (V. Chellaboina), wm.haddad@aerospace.gatech.edu (W.M. Haddad), qing_hui@ae.gatech.edu (Q. Hui), jayanthy@utk.edu (J. Ramakrishnan).
[1] Tel.: +1 865 974 5308; fax: +1 865 974 5274.
[2] Tel.: +1 404 894 3474; fax: +1 404 894 3474.
[3] Tel.: +1 865 974 6751; fax: +1 865 974 5274.

0167-6911/$ - see front matter © 2008 Elsevier B.V. All rights reserved.
doi:10.1016/j.sysconle.2008.01.008

should account for material, energy, or information in transit between compartments. Hence, to accurately describe the evolution of the aforementioned systems, it is necessary to include in any mathematical model of the system dynamics some information of the past system states. In this case, the state of the system at a given time involves a piece of trajectories in the space of continuous functions defined on an interval in the nonnegative orthant of the state space. This of course leads to (infinite-dimensional) delay dynamical systems [15,16].

Nonnegative and compartmental models are also widespread in agreement problems in dynamical networks with directed graphs and switching topologies [17,18]. Specifically, distributed decision-making for coordination of networks of dynamic agents involving information flow can be naturally captured by compartmental models. These dynamical network systems cover a very broad spectrum of applications including cooperative control of unmanned air vehicles, distributed sensor networks, swarms of air and space vehicle formations [19,20], and congestion control in communication networks [21]. In many applications involving multiagent systems, groups of agents are required to agree on certain quantities of interest. In particular, it is important to develop consensus protocols for networks of dynamic agents with directed information flow, switching network topologies, and possible system time-delays.

In this paper, we use compartmental dynamical system models to characterize dynamic algorithms for linear and nonlinear networks of dynamic agents in the presence of inter-agent communication delays that possess a continuum of *semistable* equilibria, that is, protocol algorithms that guarantee convergence to Lyapunov stable equilibria. In addition, we show that the steady-state distribution of the dynamic network is uniform, leading to system state equipartitioning or consensus. From a practical viewpoint, it is not sufficient to only guarantee that a network converges to a state of consensus since steady-state convergence is not sufficient to guarantee that small perturbations from the limiting state will lead to only small transient excursions from a state of consensus. It is also necessary to guarantee that the equilibrium states representing consensus are Lyapunov stable, and consequently, semistable. These results extend the results in the literature on consensus protocols for linear balanced networks to linear and nonlinear unbalanced networks with time-delays.

2. Mathematical preliminaries

In this section we introduce notation, several definitions, and some key results concerning linear nonnegative dynamical systems with time-delay [22,23] that are necessary for developing some of the main results of this paper. Specifically, for $x \in \mathbb{R}^n$ we write $x \geq\geq 0$ (resp., $x \gg 0$) to indicate that every component of x is nonnegative (resp., positive). In this case, we say that x is *nonnegative* or *positive*, respectively. Likewise, $A \in \mathbb{R}^{n\times m}$ is *nonnegative*[4] or *positive* if every entry of A is nonnegative or positive, respectively, which is written as $A \geq\geq 0$ or $A \gg 0$, respectively. Furthermore, for $A \in \mathbb{R}^{n\times n}$ we write $A \geq 0$ (resp., $A > 0$) to indicate that A is a nonnegative-definite (resp., positive-definite) matrix. In addition, rank(A) denotes the rank of a matrix A, spec(A) denotes the spectrum of A, $(\cdot)^{\mathrm{T}}$ denotes transpose, and $(\cdot)^{\mathrm{D}}$ denotes the Drazin generalized inverse. Recall that for a diagonal matrix $A \in \mathbb{R}^{n\times n}$ the Drazin inverse $A^{\mathrm{D}} \in \mathbb{R}^{n\times n}$ is given by $A^{\mathrm{D}}_{(i,i)} = 0$ if $A_{(i,i)} = 0$ and $A^{\mathrm{D}}_{(i,i)} = 1/A_{(i,i)}$ if $A_{(i,i)} \neq 0$, $i = 1, \ldots, n$ [24, p. 227]. Let $\overline{\mathbb{R}}^n_+$ and $\mathbb{R}^n_+$ denote the nonnegative and positive orthants of $\mathbb{R}^n$, that is, if $x \in \mathbb{R}^n$, then $x \in \overline{\mathbb{R}}^n_+$ and $x \in \mathbb{R}^n_+$ are equivalent, respectively, to $x \geq\geq 0$ and $x \gg 0$. Finally, let $\mathbf{e} \in \mathbb{R}^n$ denote the ones vector of order n, that is, $\mathbf{e} = [1, \ldots, 1]^{\mathrm{T}}$, and let $\mathbf{e}_i \in \mathbb{R}^n$ denote the elementary vector of order n with 1 in the ith location and 0's elsewhere.

The following definition introduces the notion of a nonnegative (resp., positive) function.

Definition 2.1. Let $T > 0$. A real function $x : [0, T] \to \mathbb{R}^n$ is a *nonnegative* (resp., *positive*) *function* if $x(t) \geq\geq 0$ (resp., $x(t) \gg 0$) on the interval $[0, T]$.

The next definition introduces the notion of essentially nonnegative matrices and compartmental matrices.

Definition 2.2 (*[12]*). Let $A \in \mathbb{R}^{n\times n}$. A is *essentially nonnegative* if $A_{(i,j)} \geq 0$, $i, j = 1, \ldots, n$, $i \neq j$. A is *compartmental* if A is essentially nonnegative and $A^{\mathrm{T}}\mathbf{e} \leq\leq 0$.

In the first part of this paper, we consider linear, time-delay dynamical systems $\mathcal{G}$ of the form

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^{n_{\mathrm{d}}} A_{\mathrm{d}i}x(t - \tau_i), \quad x(\theta) = \eta(\theta),$$
$$-\bar{\tau} \leq \theta \leq 0, \quad t \geq 0, \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $t \geq 0$, $A \in \mathbb{R}^{n\times n}$, $A_{\mathrm{d}i} \in \mathbb{R}^{n\times n}$, $\tau_i \in \mathbb{R}$, $i = 1, \ldots, n_{\mathrm{d}}$, $\bar{\tau} = \max_{i\in\{1,\ldots,n_{\mathrm{d}}\}}\tau_i$, $\eta(\cdot) \in \mathcal{C}_+ \triangleq \{\psi(\cdot) \in \mathcal{C}([-\bar{\tau}, 0], \mathbb{R}^n) : \psi(\theta) \geq\geq 0, \theta \in [-\bar{\tau}, 0]\}$ is a continuous vector-valued function specifying the initial state of the system, and $\mathcal{C}([-\bar{\tau}, 0], \mathbb{R}^n)$ denotes a Banach space of continuous functions mapping the interval $[-\bar{\tau}, 0]$ into $\mathbb{R}^n$ with the topology of uniform convergence. Note that the state of (1) at time t is the *piece of trajectories* x between $t - \tau$ and t, or, equivalently, the *element* x_t in the space of continuous functions defined on the interval $[-\bar{\tau}, 0]$ and taking values in $\mathbb{R}^n$, that is, $x_t \in \mathcal{C}([-\bar{\tau}, 0], \mathbb{R}^n)$, where $x_t(\theta) \triangleq x(t + \theta)$, $\theta \in [-\bar{\tau}, 0]$. Furthermore, since for a given time t the piece of the trajectories x_t is defined on $[-\bar{\tau}, 0]$, the uniform norm $|||x_t||| = \sup_{\theta\in[-\bar{\tau},0]} \|x(t+\theta)\|$, where $\|\cdot\|$ denotes the Euclidean vector norm, is used for the definitions of Lyapunov and asymptotic stability of (1). For further details, see [15,16]. In addition, note that since $\eta(\cdot)$ is continuous it follows from Theorem 2.1 of [15, p. 14] that there exists a unique solution $x(\eta)$ defined on $[-\bar{\tau}, \infty)$ that coincides with η on $[-\bar{\tau}, 0]$ and satisfies (1) for all $t \geq 0$. Finally, recall that if the positive orbit $\gamma^+(\eta(\theta))$ of (1) is bounded, then $\gamma^+(\eta(\theta))$ is *precompact* [25], that is, $\gamma^+(\eta(\theta))$

[4] In this paper it is important to distinguish between a square nonnegative (resp., positive) matrix and a nonnegative-definite (resp., positive-definite) matrix.