Available online at www.sciencedirect.com
ScienceDirect
Systems & Control Letters 57 (2008) 228–235
SYSTEMS & CONTROL LETTERS
www.elsevier.com/locate/sysconle

Stabilizability of two-dimensional linear systems via switched output feedback

K.R. Santarelli*, A. Megretski, M.A. Dahleh

Laboratory for Information and Decision Systems, Massachusetts Institute of Technology, 77 Massachusetts Avenue, Rm. 32D-758, Cambridge, MA 02139, USA

Received 10 May 2006; received in revised form 7 September 2007; accepted 8 September 2007
Available online 22 October 2007

Abstract

The problem of stabilizing a second-order SISO LTI system of the form $\dot{x} = Ax + Bu$, $y = Cx$ with feedback of the form $u(x) = v(x)Cx$ is considered, where $v(x)$ is real-valued and has domain which is all of $\mathbb{R}^2$. It is shown that, when stabilization is possible, $v(x)$ can be chosen to take on no more than two values throughout the entire state-space (i.e., $v(x) \in \{k_1, k_2\}$ for all x and for some k_1, k_2), and an algorithm for finding a specific choice of $v(x)$ is presented. It is also shown that the classical root locus of the corresponding transfer function $C(sI - A)^{-1}B$ has a strong connection to this stabilization problem, and its utility is demonstrated through examples.

Keywords: Switched systems; Hybrid systems; Asymptotic stability; Root locus; Output feedback

1. Introduction

The study of hybrid systems is an area that has pervaded research for more than a decade (see, e.g., [2–6,8,9,11,13,14]). In particular, stabilization of continuous time systems via hybrid feedback is a problem that has received much attention in the recent literature. Artstein first addressed this question through examples [1]. Litsyn et al. show in [10] that the linear system

$$\dot{x} = Ax + Bu, \quad y = Cx \tag{1}$$

with (A, B) reachable and (C, A) observable can be stabilized via a hybrid feedback controller which uses a countable number of discrete states (and no continuous states) and which only depends upon the output y as opposed to the entire continuous state x. A natural question arises as to whether a hybrid feedback controller can be designed which uses a *finite* number of states instead. For the most part, the answer to this question is still open, though a partial answer has been given by Hu et al. in [7] based upon the so-called conic switching laws of [15,16]. In [7], it is shown that, for a certain class of single-input, single-output (SISO) second-order systems which are reachable and observable, there exists a feedback control law of the form $u(x) = v(x)Cx$ where

$$v(x) = \begin{cases} k_1 & \text{if } x_1x_2 \geqslant 0, \\ k_2 & \text{if } x_1x_2 < 0 \end{cases} \tag{2}$$

with $x = [x_1 \;\; x_2]'$ such that the resulting closed-loop system

$$\dot{x} = Ax + v(x)BCx \tag{3}$$

is globally exponentially stable. A control law of the form (2) is desirable as it can be implemented as a switch between two static gains which multiplies the output $y = Cx$. Note that, in general, the above strategy does not always work as the result of [10] sometimes requires a more complicated hybrid feedback structure to achieve stability, even when the system described by (1) is reachable and observable.

Example 1.1. Consider (1) with

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = [0 \quad 1].$$

This system is reachable and observable, but (3) is not stable for *any* real-valued choice of $v(x) \equiv v(x_1, x_2)$, not just $v(x_1, x_2)$ of

* Corresponding author. Tel.: +1 617 253 3449; fax: +1 617 258 5779.
E-mail addresses: krsanta@mit.edu (K.R. Santarelli), ameg@mit.edu (A. Megretski), dahleh@mit.edu (M.A. Dahleh).

0167-6911/$ - see front matter © 2007 Elsevier B.V. All rights reserved.
doi:10.1016/j.sysconle.2007.09.003

the form (2). To see this, first note that the region $x_1 < 0, x_2 > 0$ is invariant under the flow of (3) for any choice of $v(x_1, x_2)$. Indeed, when $x_1 = 0$, $\dot{x}_1 = -x_2 < 0$, and when $x_2 = 0$, $\dot{x}_2 = -x_1 > 0$ for all choices of $v(x)$. Moreover, when $x_1(0) < 0$ and $x_2(0) > 0$, $\dot{x}_1 = 2x_1 - x_2 < 0$, which means that $x_1(t)$ is strictly decreasing, and, hence, does not decay to zero regardless of the choice of $v(x_1, x_2)$.

The goal of this paper is to answer the following questions: under what conditions on $A \in \mathbb{R}^{2\times 2}$, $B \in \mathbb{R}^{2\times 1}$ and $C \in \mathbb{R}^{1\times 2}$ can the closed-loop system (3) be made asymptotically stable for some choice of $v(x_1, x_2)$? And, moreover, when stability is achievable, how may one design $v(x_1, x_2)$ explicitly? As it turns out, the answer to the first question has a strong connection to the classical control notion of root locus. Essentially, if one considers control laws of the form $v(x_1, x_2)=k$ for some $k \in \mathbb{R}$, then the system (3) is stabilizable in only one of two situations:

- There exists a value of k such that the matrix $A + kBC$ is Hurwitz and, hence, (3) is exponentially stabilizable via static output feedback.
- There is no value of k for which $A + kBC$ is Hurwitz, but there does exist a value of k for which the eigenvalues of $A + kBC$ are complex. In this case, $v(x_1, x_2)$ can be chosen to take on only two values k_1 and k_2 throughout the entire state-space, i.e., $v(x_1, x_2) \in \{k_1, k_2\}$, where k_1 and k_2 are appropriately selected real constants, and global exponential stability can be achieved.

A third situation can exist in which there exists no value of k for which $A + kBC$ is Hurwitz and the eigenvalues of $A + kBC$ are real for all k. It is precisely these situations for which no choice of $v(x_1, x_2)$ will yield asymptotic stability.

Note that, unlike [10], the switching strategies employed here and in [7] in general require full knowledge of the state x of (1) rather than just knowing the output $y = Cx$. While we will not formally show this here, an appropriate first-order LTI observer of the plant state x can be designed to implement a slight variant of the control laws we discuss here (see [12] for details of this work). The work we discuss here is a necessary precursor to this more general problem, much like the linear system pole placement problem via state feedback is a precursor to the pole placement problem via output feedback.

The structure of the paper is as follows. First, we examine two particular case studies in which the form of the B and C vectors have special structure and analyze the conditions on the matrix A which will guarantee stability. Also, we will derive explicit forms for $v(x_1, x_2)$ which can be used to achieve stability when it is possible to do so. Next, we will show that, through appropriate coordinate transformations, all nontrivial[1] problems can be transformed into either one of these two case studies and then will use this to establish the main result. Finally, we explore a general method of designing such controllers (when they exist) and provide several examples to illustrate the methodology.

[1] By "nontrivial", we refer to problems in which neither B nor C is identically 0.

2. Case studies

In this section, we explore two specific case studies in which the A, B, and C matrices of (1) have particular structures. Using appropriate coordinate transformations, we will then relate the results of this section to derive the main result for general A, B, and C.

2.1. Case 1

We first assume a system of the following structure:

$$A = \begin{bmatrix} a & b \\ c & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = [0 \quad 1], \tag{4}$$

where $a, c \in \mathbb{R}$, and $b \geqslant 0$. Here, (3) takes the form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & v(x_1, x_2) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{5}$$

We summarize the possibilities for stabilizability as a function of the parameters a, b, and c in the proposition below:

Proposition 2.1. *For system* (5):

(1) *If $bc = 0$, then* (5) *is exponentially stabilizable via static output feedback if $a < 0$ and is not stabilizable for any choice of $v(x_1, x_2)$ otherwise.*
(2) *If $b > 0$ and $c > 0$, when $v(x_1, x_2) = k$ for some constant k, then the eigenvalues of* (5) *are real for all k, and* (5) *is either exponentially stabilizable via static output feedback or is not stabilizable by any choice of $v(x_1, x_2)$.*
(3) *If $b > 0$ and $c < 0$, when $v(x_1, x_2) = k$ for some constant k, then the eigenvalues of* (5) *are not real for all k, and* (5) *is exponentially stabilizable either by static output feedback or by feedback of the form*

$$v(x_1, x_2) = \begin{cases} k_1 & \text{if } w_1' x = 0, \\ k_2 & \text{if } w_1' x \neq 0 \end{cases}$$

for some appropriate choice of w_1, k_1, and k_2.

We prove each part separately below.

Proof of Part 1. Note that if $b = 0$, the system described by (4) has an uncontrollable mode. In this case, stabilizability is possible if and only if $a < 0$ and can be achieved via $v(x_1, x_2) = k$, where $k < 0$. In a similar vein, if $c=0$, (4) has an unobservable mode. Noting that any initial condition with $x_2(0) = 0$ satisfies $x_2(t) = 0$ for all t, it is again clear that stabilizability is possible if and only if $a < 0$ and can be achieved by setting $v(x_1, x_2)$ to a negative real constant.

Proof of Part 2. If we set $v(x_1, x_2) = k$ for some constant k, the characteristic polynomial of (5) is given by

$$s^2 - (a + k)s + ak - bc. \tag{6}$$