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The consensus problem in the behavioral approach

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ABSTRACT

In this paper we investigate the multi-agent consensus problem in a broad context, by assuming both for the agents and for the distributed controllers higher order input–output dynamic models.

The behavioral approach developed by Jan Willems seems to be the most appropriate set-up where to investigate this general problem.

By making use of the behavioral approach, we will show that the consensus problem can be naturally rephrased as a special case of stabilization problem: the stabilization pertains only a part of the system variables (the outputs) and it is achieved through regular full interconnection of the agent models and of the controllers. It turns out that if the communication among agents is described by a weighted, undirected and connected graph, then a necessary and sufficient condition for the consensus problem to be solvable is that the output is stabilizable from the input in the agents model. In this respect, the theory here developed for higher-order input–output models naturally extends the results about consensus derived in the state-space approach.

To Jan: a mentor and a friend. Amazingly inspiring and entertaining in both roles.

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1. Introduction

The mathematical formulation of multi-agents systems and consensus problems was introduced several years ago in some pioneering papers such as [1–3]. But it was only a decade ago that a wide stream of research on these topics started, thanks to milestone contributions such as [4–8]. Aside from the theoretical challenges that these problems pose, strong motivations for such a widespread interest come from the numerous application problems that can be naturally stated as consensus problems. Indeed, when dealing with sensor networks, coordination of mobile robots or UAVs, flocking and swarming in animal groups, dynamics of opinion forming, etc., the main control target can be mathematically formalized as a consensus problem among agents, exchanging information and resorting to distributed algorithms that make use of the information collected from neighboring agents (see, e.g. [9,10]).

While the first contributions on this subject focused on agents described as simple or double integrators, more recent works addressed the case of agents described by higher order models [4,10–13]. The vast majority of the literature on consensus, however, assume that the homogeneous agents dynamics is described by a state-space model and that consensus is achieved through a static state- or output-feedback, that makes use of the weighted information collected from the neighboring agents, in a cooperative set-up (see [14] for consensus under antagonistic interactions).

The aim of this paper is to investigate the multi-agent consensus problem in a broader context, by assuming both for the agents and for the distributed controllers higher order input/output dynamic models. The behavioral approach developed by Jan Willems [15–17] seems to be a convenient set-up where to investigate this general problem. Since, to the best of our knowledge, this set-up has never been used before in this context, we have tried to make the paper as self-contained as possible, by recalling the few fundamental definitions and results that are necessary to understand the technical details of the paper. A comprehensive treatment of the behavior theory can be found in any of the three aforementioned references.

By making use of the behavioral approach, we will show that the consensus problem can be naturally rephrased as a variant of the stabilization problem: the stabilization pertains only to a part of the system variables (the outputs) and it is achieved through regular full interconnection of the agents models and of the controllers. We will prove that if the communication among agents

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is described by a weighted, undirected and connected graph, then a necessary and sufficient condition for the consensus problem to be solvable is that the output is stabilizable from the input in the agents model. In this respect, the theory here developed for higher-order input/output models naturally extends the results about consensus derived in the state-space approach (see [12], for instance).

The paper is organized as follows. In Section 2 preliminary definitions, notation and results are given. In Section 3 the consensus problem is posed. Section 4 provides a characterization of the controllers that make it possible for the agents to achieve consensus and Section 5 provides a similar characterization under the additional assumption that the consensus is achieved by means of a regular interconnection. Section 6 provides a complete solution to the consensus problem. Section 7 concludes the paper by showing how the most classical result on consensus for agents described by state-space models easily follows from the present analysis.

A preliminary version of the results appearing in Sections 3–5 of this paper has appeared, in a more general set-up, in [18]. However, no problem solution was provided: a characterization of the controllers that solve the problem was given, but no necessary and sufficient condition for the existence of such controllers, and hence for the consensus problem solvability, was provided.

2. Preliminaries

We introduce some notation and definitions that will be used in the following.

I_p denotes the $p \times p$ *identity matrix*. The p-dimensional vector with all entries equal to 1 is denoted by $\mathbf{1}_p$, while the ith *standard basis vector* in $\mathbb{R}^p$ (also known as the ith canonical vector) is denoted by e_i. The *spectrum* of a square matrix L is denoted by $\sigma(L)$. $\text{diag}\{v_1, v_2, \ldots, v_p\}$ is the $p \times p$ diagonal matrix with diagonal entries $v_1, v_2, \ldots, v_p$.

We let $\mathbb{R}[s]$ denote the ring of polynomials in the indeterminate s with real coefficients. A polynomial $p \in \mathbb{R}[s]$ is *Hurwitz* if all its zeros belong to $\{s \in \mathbb{C} : \text{Re}(s) < 0\}$. A polynomial matrix $P = P(s) \in \mathbb{R}[s]^{p\times q}$ is *right prime* if it is of full column rank q and the greatest common divisor of its maximal order minors is a unit, equivalently if $\text{rank}\, P(\lambda) = q$ for every $\lambda \in \mathbb{C}$. It is well-known [19] that $P(s)$ is right prime if and only if it admits a polynomial left inverse or, equivalently, the Bézout equation

$$XP = I_q$$

in the unknown polynomial matrix $X(s) \in \mathbb{R}[s]^{q\times p}$ is solvable. *Left prime* matrices are similarly defined and characterized. A square and nonsingular polynomial matrix $P = P(s) \in \mathbb{R}[s]^{q\times q}$ whose inverse P^{-1} is polynomial is called *unimodular*. Clearly a unimodular matrix is both right prime and left prime.

Every polynomial matrix $P \in \mathbb{R}[s]^{p\times q}$ of rank r factorizes over $\mathbb{R}[s]$ as $P = L\Delta R$, where L is $p \times r$ and right prime, Δ is $r \times r$ and nonsingular, and R is $r \times q$ and left prime.

The concepts of *left annihilator* and, in particular, of *minimal left annihilator* (MLA, for short) of a given polynomial matrix P have been originally introduced in [20] and can be summarized as follows: if P is a $p \times q$ polynomial matrix of rank r, a polynomial matrix H is a left annihilator of P if $HP = 0$. A left annihilator H_m of P is an MLA if it is of full row rank and for any other left annihilator H of P we have $H = QH_m$ for some polynomial matrix Q. It can be easily proved that, unless P is of full row rank, an MLA always exists (if P is of full row rank, its left annihilators are zero matrices with an arbitrary number of rows), it is a $(p - r) \times p$ left prime matrix and is uniquely determined modulo a unimodular left factor.

In the paper we consider (continuous-time) signals defined on the *time set* $\mathbb{R}$. Signals will be real valued and hence they will be, in general, elements of $(\mathbb{R}^q)^{\mathbb{R}}$, for some $q \in \mathbb{N}$. By $\mathscr{F}^q$ we will denote the set of arbitrarily often differentiable functions, i.e., $\mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^q) \subseteq (\mathbb{R}^q)^{\mathbb{R}}$.

For every $P = \sum_{i=0}^n P_i s^i \in \mathbb{R}[s]^{p\times q}$, we associate with P the polynomial matrix differential operator $P\circ = \sum_{i=0}^n P_i \frac{d^i}{dt^i}$. The action of such a polynomial matrix differential operator P on any signal $w \in \mathscr{F}^q$ is denoted by $P \circ w$.

In this paper by a *system* we mean a triple $\Sigma = (\mathbb{R}, \mathbb{R}^q, \mathscr{B})$, where $\mathbb{R}$ is the time set, $\mathbb{R}^q$ is the set where the system trajectories take values, and $\mathscr{B}$ is the *behavior*, namely the set of admissible trajectories of the system variable w. We will consider linear, time-invariant behaviors described as the kernels of polynomial matrix operators. This means that there exists a polynomial matrix $P \in \mathbb{R}[s]^{k\times q}$ such that

$$\mathscr{B} = \{w \in \mathscr{F}^q : P \circ w = 0\}. \quad (1)$$

It is always possible to find a matrix $\overline{P} \in \mathbb{R}[s]^{r\times q}$ of full row rank r such that $\mathscr{B} = \{w \in \mathscr{F}^q : \overline{P} \circ w = 0\}$.

A behavior $\mathscr{B} \subseteq \mathscr{F}^q$ is *autonomous* if it is a finite dimensional vector subspace of $\mathscr{F}^q$ as a vector space on $\mathbb{R}$. $\mathscr{B}$ described as in (1) is autonomous if and only if $P \in \mathbb{R}[s]^{k\times q}$ is of full column rank q.

An autonomous behavior (1) is *stable* if the greatest common divisor of the maximal (i.e., qth) order minors of P is a Hurwitz polynomial. If P is of full row rank and hence, under the autonomy assumption, square and nonsingular, this amounts to requiring that $\det P$ is Hurwitz. A trajectory $w \in \mathscr{F}^q$ is called *small* if it belongs to some stable autonomous behavior or, equivalently, if it satisfies the equation $p \circ w = 0$ for some Hurwitz polynomial p. Clearly, small signals are the polynomial–exponential functions that converge to zero as the time approaches $+\infty$.

If we partition the system variables as $w = \begin{pmatrix} y \\ u \end{pmatrix} \in \mathscr{F}^{p+m}$, and accordingly describe the behavior $\mathscr{B}$ as

$$\mathscr{B} = \left\{ \begin{pmatrix} y \\ u \end{pmatrix} \in \mathscr{F}^{p+m} : P_y \circ y = P_u \circ u \right\},$$

$$(P_y \;\; -P_u) \in \mathbb{R}[s]^{k\times(p+m)},$$

we say that u is *free* in $\mathscr{B}$ if for any $u \in \mathscr{F}^m$ there exists $y \in \mathscr{F}^p$ such that $\begin{pmatrix} y \\ u \end{pmatrix} \in \mathscr{B}$. This is the case if and only if $\text{rank}(P_y \;\; -P_u) = \text{rank}(P_y)$. If additionally the behavior

$$\mathscr{B}^0 = \left\{ y \in \mathscr{F}^p : P_y \circ y = 0 \right\}$$

is autonomous, we say that $\mathscr{B}$ is an *input/output behavior* with *input* u and *output* y. Clearly, this is the case if and only if $\text{rank}(P_y \; -P_u) = \text{rank}(P_y) = p$. If the matrix $(P_y \; -P_u)$ is of full row rank k, it follows that $\mathscr{B}$ is an input/output behavior with input u and output y if and only if $k = p$ and P_y is nonsingular.

If $\mathscr{B}$ and $\mathscr{C}$ are behaviors in $\mathscr{F}^q$, described as kernels of the polynomial matrix operators $P\circ$ and $C\circ$, respectively, we denote the *interconnection of* $\mathscr{B}$ *and* $\mathscr{C}$ as follows:

$$\mathscr{B} \wedge \mathscr{C} := \left\{ w \in \mathscr{F}^q : w \in \mathscr{B}, w \in \mathscr{C} \right\} = \left\{ w \in \mathscr{F}^q : \begin{pmatrix} P \\ C \end{pmatrix} \circ w \right\}.$$

The interconnection of $\mathscr{B}$ and $\mathscr{C}$ is said to be *regular* if

$$\text{rank} \begin{pmatrix} P \\ C \end{pmatrix} = \text{rank}(P) + \text{rank}(C).$$

If $\mathscr{B}$ is an input/output behavior, with input u and output y, and the interconnection of $\mathscr{B}$ and $\mathscr{C}$ is regular, the input/output structure of $\mathscr{B}$ is preserved even after interconnection: this means that it is still possible to add (free) signals u' to the components of u after interconnection [21]. More precisely, the components of u' are free in the behavior $\mathscr{B} \wedge \mathscr{C}$, illustrated in Fig. 1.