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Multidimensional behaviors: The state-space paradigm[☆]

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ABSTRACT

For 1-D systems, the state-space approach has perhaps become the most popular method of analyzing these systems. Central to the idea of the state-space approach is the ability to write the system equations in a first-order form by introducing new variables called the state variables. There have been several attempts to imitate the state-space framework for n-D systems. Introduction of behavioral theory by Jan C. Willems, has given fresh impetus to this attempt to imitate state-space framework for n-D systems. In this paper, dedicated to Jan Willems, we provide our recent attempt at obtaining a state-space framework for n-D systems.

1. Introduction

Jan C. Willems, in his celebrated paper *Paradigms and puzzles in the theory of dynamical systems* [1], wrote: "In engineering, particularly in control and signal processing, there has always been a tendency to view systems as *processors*, producing output signals from input signals. In many applications in control engineering and signal processing, it will, indeed, be eminently clear what the inputs and the outputs are. However, there are also many applications where this input–output structure is not at all evident (an example at point is in the terminal behavior of an electrical circuit)". Willems pointed out a number of situations where it is indeed impractical to assume an input/output structure on the system variables. Examples include Kepler's laws of planetary motion, econometrics [2], economics (relation between production, capital cost and labor cost) and discrete event systems [1]. Willems also argued that there are dynamical systems (e.g., *Leontief economy* [2]), for which it is outright *impossible* to obtain an input/output model. In a series of works [1–6] Willems brought about a radical change in the way a mathematical model for dynamical systems should be viewed. He showed that systems viewed as maps from inputs (plus initial conditions) to outputs is perhaps not the most suited approach as a modeling paradigm for dynamical systems. There is in fact a more fundamental object, the *behavior* of the system – that is, the collection of trajectories allowed by the laws of physics applied to the system – that provides a better mathematical model for dynamical systems. With this rudimentary object – the behavior – Willems succeeded in deducing – and refining where required – several existing notions of dynamical systems: linearity, shift/time-invariance, input/output representation, autonomy, controllability, observability, stability, stabilizability, detectability, state-variables, etc. This was a remarkable feat, a *paradigm shift*, indeed. This approach became known as the *behavioral approach* to systems theory.

The limitations of the input/output approach become much more exposed for *multidimensional systems* (also called *n D systems*) — that is, dynamical systems with more than one independent variables. For example, consider the system described by the following partial differential equation (PDE):

$$\frac{\partial w_1}{\partial x_1} - \frac{\partial w_2}{\partial x_2} = 0.$$

Although one may view w_2 as an input and w_1 as an output, modeling the system as a mapping from w_2 to w_1 is fraught with technical problems. (For instance, the 'transfer function' here would be $\frac{s_2}{s_1}$, which has numerator and denominator sharing a common root! See [7, Remark 76] for a more elaborate discussion on this.) However, the behavior (that is, the set of solutions of the above equation) still exists, and hence, many system-theoretic questions posed in behavioral approach of 1D systems would make perfect sense for this system too. Spurred by Willems' treatment of 1D systems, issues like autonomy, controllability, observability, stability for nD systems were tackled and resolved using the behavioral approach (see [8] for $n = 2$, and [7,9,10] among others for general n). With this development, the behavioral approach has

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become one of the strongest contenders for a *grand unified theory* of systems and control.

Despite the success of the behavioral approach to nD systems, the current state-of-the-art lags far behind its 1D counterpart. On several issues, for which there has been a well-accepted solution in 1D systems, for nD systems a successful resolution has either completely evaded the community, or there have been many resolutions none of which were universally acceptable. Defining state-variables and obtaining state-space representations from a given representation of an nD system is one such issue that is still largely open. In this paper, we hope to provide a partial answer to this issue. State-space representation of nD systems has been an active field of research, especially for $n = 2$; see [11–14,8] among many notable works. However, these works suffer from a crucial drawback: their restricted applicability. Indeed, in each of the earlier works, several restrictive assumptions were made. For example, in [12–14] that deal with state-space models for discrete 2D systems, the systems concerned are assumed to satisfy a certain notion of *causality* in 2D integer grid. Needless to say, many 2D systems do not satisfy this assumption. In this paper, we provide a methodology to construct state-space and a first order evolution law for general nD systems that are described by linear partial differential/difference equations with constant real coefficients; we make only one assumption: the system is autonomous.

2. Background

2.1. nD systems

Following Willems, we define a dynamical system by a triplet $(\mathbb{T}, \mathbb{W}, \mathfrak{B})$, where $\mathbb{T}$ is the *indexing set* (the set of independent variables over which the system's variables, w, evolve), $\mathbb{W}$ is the *signal space* (the set from where the manifest variables take values), and $\mathfrak{B}$ is the *behavior* of the system (the subset of the set of all possible trajectories, $\mathbb{W}^{\mathbb{T}}$, that are allowed by the system). In this paper, we shall assume $\mathbb{W} = \mathbb{R}^{\mathtt{w}}$; $\mathtt{w}$ denotes the cardinality of the vector w. These variables w are called *manifest variables*. *Multidimensional (n D) systems* are characterized by the fact that they have n independent variables; that is, the indexing set $\mathbb{T}$ is either $\mathbb{Z}^n$ or $\mathbb{R}^n$. We shall use the term *continuous or discrete n D systems* for the case when $\mathbb{T} = \mathbb{R}^n$ or $\mathbb{T} = \mathbb{Z}^n$, respectively. The letter $\boldsymbol{t}$ will be used to denote the independent variable; that is, $\boldsymbol{t} \in \mathbb{R}^n$ for continuous systems, and $\boldsymbol{t} \in \mathbb{Z}^n$ for discrete systems. In this paper, we are going to look at a special kind of nD systems, namely, systems that are described by linear partial differential/difference equations with constant real coefficients.

Behaviors of continuous nD systems are sets of solutions to partial differential equations (PDEs). Such PDEs are written using polynomials in partial differential operators. Let ∂_i denote the partial differential operator with respect to the variable t_i, that is, $\partial_i = \frac{\partial}{\partial t_i}$. The polynomial ring in the variables $\{\partial_1, \ldots, \partial_n\}$ is denoted by $\mathbb{R}[\partial_1, \ldots, \partial_n]$. We often use the short-hand ∂ to denote the n-tuple $\{\partial_1, \ldots, \partial_n\}$. The idea of solutions of PDEs intrinsically depends on the function space where solutions are sought. In this paper, we shall consider the space of *smooth functions*, denoted by $\mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}})$. Thus, a behavior $\mathfrak{B}$ of a continuous nD system, described by a set of linear partial differential equations with constant real coefficients, can be defined as

$$\mathfrak{B} := \left\{ w \in \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}}) \mid R(\partial) w = 0 \right\}, \tag{1}$$

where $R(\partial) \in \mathbb{R}^{\bullet\times\mathtt{w}}[\partial]$. For obvious reasons, Eq. (1) is called a *kernel representation* of $\mathfrak{B}$ and $R(\partial)$ is called a *kernel representation matrix* of $\mathfrak{B}$. We write $\mathfrak{B} = \ker R(\partial)$ for brevity.

For discrete nD systems, the role of ∂_is is played by the shift operators, σ_is. In this case, the function space that we consider is the space of vector valued ($\mathtt{w}$ tuple) sequences indexed by $\mathbb{Z}^n$, i.e., $(\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}^n} = \{w : \mathbb{Z}^n \to \mathbb{R}^{\mathtt{w}}\}$. In this paper, we use the symbol $\mathcal{W}(\mathbb{Z}^n, \mathbb{R}^{\mathtt{w}})$ to denote this space. The i^{th} shift operator σ_i acts on a discrete trajectory $w \in \mathcal{W}(\mathbb{Z}^n, \mathbb{R}^{\mathtt{w}})$ as

$$(\sigma_i w)(t_1, \ldots, t_n) = w(t_1, \ldots, t_i + 1, \ldots, t_n), \tag{2}$$

for all $(t_1, \ldots, t_n) \in \mathbb{Z}^n$. Note that σ_i^{-1} is a legitimate operator on $\mathcal{W}(\mathbb{Z}^n, \mathbb{R}^{\mathtt{w}})$. Thus, unlike the continuous case, the operator algebra for the discrete case contains polynomials having terms with (finite) positive as well as (finite) negative powers. Therefore, the operator algebra, in this case, is given by the *Laurent polynomial ring* in the variables $\{\sigma_1, \ldots, \sigma_n\}$. We denote this ring by $\mathbb{R}[\sigma_1, \sigma_1^{-1}, \ldots, \sigma_n, \sigma_n^{-1}]$. Like in the case of partial differential operators, we shall use the singleton σ to denote the n-tuple $\{\sigma_1, \ldots, \sigma_n\}$, and, likewise, we shall write $\mathbb{R}[\sigma, \sigma^{-1}]$ to denote the n-variable Laurent polynomial ring in the shifts $\{\sigma_1, \ldots, \sigma_n\}$. Consequently, a behavior $\mathfrak{B}$ of a discrete nD system, which is the solution set of a system of partial difference equations gets defined as

$$\mathfrak{B} := \left\{ w \in \mathcal{W}(\mathbb{Z}^n, \mathbb{R}^{\mathtt{w}}) \mid R(\sigma) w = 0 \right\}, \tag{3}$$

where $R(\sigma) \in \mathbb{R}^{\bullet\times\mathtt{w}}[\sigma, \sigma^{-1}]$. As in the continuous case, Eq. (3) is called a *kernel representation* of $\mathfrak{B}$ and $R(\sigma)$ is called a *kernel representation matrix*, while $\mathfrak{B}$ is written in short as $\mathfrak{B} = \ker R(\sigma)$.

It is apparent from the last two paragraphs that continuous and discrete systems share a common model of description – the kernel representation – with only the operator algebras and the function spaces being different. This commonality is utilized throughout this paper — so much so that we use common symbols to denote various objects that relate to both discrete and continuous systems. For example, $\mathcal{A}$ is the *operator algebra* ($\mathcal{A} = \mathbb{R}[\partial]$ (continuous), or $\mathbb{R}[\sigma, \sigma^{-1}]$ (discrete)), $\mathcal{F}_n^{\mathtt{w}}$ is the *function space* ($\mathcal{F}_n^{\mathtt{w}} = \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}})$ (continuous), or $\mathcal{W}(\mathbb{Z}^n, \mathbb{R}^{\mathtt{w}})$ (discrete)), ξ is the n-tuple of operators ($\xi = \partial$ (continuous), or σ (discrete)). We shall use ξ^ν to denote the monomial $\xi_1^{\nu_1} \cdots \xi_n^{\nu_n}$, where $\nu = (\nu_1, \ldots, \nu_n) \in \mathbb{Z}^n$. The collection of all nD systems that have $\mathtt{w}$ manifest variables and are described by linear partial differential/difference equations is denoted by $\mathfrak{L}_n^{\mathtt{w}}$. We often abuse this notation and write $\mathfrak{B} \in \mathfrak{L}_n^{\mathtt{w}}$. Thus, a continuous/discrete behavior $\mathfrak{B} \in \mathfrak{L}_n^{\mathtt{w}}$ is described as $\mathfrak{B} = \ker R(\xi) \subseteq \mathcal{F}_n^{\mathtt{w}}$, where $R(\xi) \in \mathcal{A}^{\bullet\times\mathtt{w}}$. Another representation of $\mathfrak{B} \in \mathfrak{L}_n^{\mathtt{w}}$, called a *latent variable representation*, is required in the sequel. In this representation, $\mathfrak{B} \in \mathfrak{L}_n^{\mathtt{w}}$ is described as

$$\mathfrak{B} := \left\{ w \in \mathcal{F}_n^{\mathtt{w}} \mid \exists \ell \in \mathcal{F}_n^{r} \text{ such that } R(\xi) w = M(\xi) \ell \right\}, \tag{4}$$

where $R(\xi) \in \mathcal{A}^{g\times\mathtt{w}}$ and $M(\xi) \in \mathcal{A}^{g\times r}$. The variables ℓ are called *latent variables*.

2.2. The equation module and the quotient module

Following the trend set by Willems for 1D systems, in this paper, we treat nD systems algebraically via the operator algebra $\mathcal{A}$. In this connection, two algebraic objects are of great importance to the analysis. The first one of these algebraic objects is called the *equation module*, denoted by $\mathcal{R}$, and is defined as follows: suppose $\mathfrak{B} \in \mathfrak{L}_n^{\mathtt{w}}$ is given by a kernel representation as $\mathfrak{B} = \ker R(\xi)$, with $R(\xi) \in \mathcal{A}^{\bullet\times\mathtt{w}}$, then $\mathcal{R}$ is defined to be the set of all (row-)vectors in $\mathcal{A}^{1\times\mathtt{w}}$ that can be written as linear combinations of the rows of $R(\xi)$ with coefficients from $\mathcal{A}$. This $\mathcal{R}$ is a submodule of the $\mathcal{A}$-module $\mathcal{A}^{1\times\mathtt{w}}$. With $\mathcal{R}$ in place, it is easy to see that $\mathfrak{B} = \ker R(\xi)$ admits an alternative description given by

$$\mathfrak{B} = \left\{ w \in \mathcal{F}_n^{\mathtt{w}} \mid r(\xi) w = 0 \text{ for all } r(\xi) \in \mathcal{R} \right\}. \tag{5}$$

Thus, given a submodule $\mathcal{R} \subseteq \mathcal{A}^{1\times\mathtt{w}}$, we can define the corresponding behavior $\mathfrak{B}(\mathcal{R})$ by Eq. (5). The following proposition gives a characterization of the equation module. The complete