Systems & Control Letters 95 (2016) 62–69

Contents lists available at ScienceDirect

Systems & Control Letters

journal homepage: www.elsevier.com/locate/sysconle

Robust synchronization of coprime factor perturbed networks

H.J. Jongsma, H.L. Trentelman *, M.K. Camlibel

Johann Bernoulli Institute for Mathematics and Computer Science, University of Groningen, Groningen, The Netherlands

ARTICLE INFO

Article history:
Available online 15 February 2016

Keywords:
Synchronization
Linear multi-agent system
Robust control
Algebraic Riccati equation

ABSTRACT

This paper deals with robust synchronization of directed and undirected multi-agent networks with uncertain agent dynamics. Given a network with identical nominal dynamics, we allow uncertainty in the form of coprime factor perturbations of the transfer matrix of the agent dynamics. These perturbations are assumed to be stable and have $\mathcal{H}_\infty$-norm that is bounded by an a priori given desired tolerance. We derive state space equations for dynamic observer based protocols that achieve robust synchronization in the presence of such uncertainty. We obtain an achievable interval, i.e. an interval such that for each value of the tolerance contained in this interval there exists a robustly synchronizing protocol.

1. Introduction

It has been generally recognized that the work of Jan Willems on dissipativity, the algebraic Riccati equation and linear matrix inequalities forms the foundation of the state space approach to $\mathcal{H}_\infty$ and robust control theory for linear systems as it developed in the eighties of the twentieth century. An example of work where many concepts put forward by Jan Willems fall into place is the work on optimal robust stabilization by Glover in [1] and Glover and MacFarlane in [2], see also [3]. In the present paper we will adopt ideas from [2] to formulate and resolve a control synthesis problem in the more recent context of robust synchronization of networked multi-agent systems. In the last decade, extensive effort has been invested in the theory of distributed control of networked multi-agent systems. Well-known problems in the theory of networked systems are the problems of *consensus* and *synchronization*, see [4–7] and [8] or, more recently, [9] and [10]. In these problems, the goal is to reach a state of agreement on certain quantities of interest which depend on the states of each agent. This is to be achieved by means of local information exchange only. A communication protocol that achieves this goal is said to achieve consensus or synchronization within the network.

Recently, in [11], results on synchronization of linear multi-agent systems have been extended to accommodate the presence of *uncertainty* in the agent dynamics. While the agents in the network have identical nominal dynamics, the actual dynamics of each agent is uncertain in the sense that the transfer matrix of each agent is a perturbation of the common nominal dynamics. In [11], *additively perturbed* agent dynamics was considered, and conditions for the existence of dynamic protocols that achieve robust synchronization and methods to obtain such protocols were established. In the current paper, we extend these results to directed and undirected networked multi-agent systems with *coprime factor perturbed* agent dynamics. We provide explicit equations for dynamic protocols that achieve robust synchronization for this kind of perturbations.

The outline of this paper is as follows. In Sections 2 and 3 we introduce some notation and review some basic facts. Next, in Section 4, the theory of synchronization of unperturbed linear multi-agent systems is briefly reviewed. Then, in Section 5 we provide a formulation of the problem of robust synchronization of coprime factor perturbed multi-agent systems. Finally in Section 6 we formulate the main results of this paper.

2. Preliminaries

In this paper, we denote the set of all proper and stable real rational matrices by $\mathcal{RH}_\infty$. If $G \in \mathcal{RH}_\infty$, then $\|G\|_\infty$ denotes its $\mathcal{H}_\infty$-norm, $\|G\|_\infty = \sup_{\mathrm{Re}(\lambda)\geq 0}\|G(\lambda)\|$. For a given square complex matrix M we denote its spectral radius by $\rho(M)$. A square matrix M is called Hurwitz if all its eigenvalues have strictly negative real parts.

Let $\mathbb{R}$ denote the field of real numbers, $\mathbb{R}^n$ the n-dimensional Euclidean space and $\mathbb{R}^{n\times n}$ the space of $n \times n$ real matrices. Denote the field of complex numbers by $\mathbb{C}$. We denote by I_p the identity matrix of dimension p and by I any identity matrix of appropriate dimension. The *Kronecker product* of the matrices $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{p\times q}$ is denoted by $A \otimes B$.

* Corresponding author.
E-mail address: h.l.trentelman@rug.nl (H.L. Trentelman).

http://dx.doi.org/10.1016/j.sysconle.2015.10.017
0167-6911/

This paper will use ideas and results from $\mathcal{H}_\infty$-control. The $\mathcal{H}_\infty$-control problem dates back to work by G. Zames in [12] and the first full solution in a state space setting was provided in [13]. A result that is instrumental in $\mathcal{H}_\infty$-control is the *bounded real lemma*, see also [3], Section 12.6.2. In this paper we will use a version of this lemma adapted to our purposes. The proof is omitted.

Lemma 1. *Consider the system $\dot{x} = Ax + Bu$, $y = Cx + Du$ with transfer matrix $G(s) = C(sI - A)^{-1}B + D$. Assume $D^TD = I$ and A is Hurwitz. Let $\tau > 1$. The $\mathcal{H}_\infty$-norm $\|G\|_\infty$ of the transfer matrix from u to y satisfies $\|G\|_\infty < \tau$ if there exists $\epsilon > 0$ and a real symmetric solution P to the Riccati inequality*

$$A^TP + PA + C^TC + \frac{1}{\tau^2 - 1}(PB + C^TD)(B^TP + D^TC) \leq -\epsilon(PB + C^TD)(B^TP + D^TC). \tag{1}$$

3. Graphs

In this paper, we consider networks whose interaction topologies are represented by directed or undirected graphs, see [14,15]. A directed graph consists of a pair $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, 2, \ldots, p\}$ is the set of *nodes*, and where $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ is the set of *edges*. Given two nodes $i, j \in \mathcal{V}$ with $i \neq j$, then an edge from i to j is represented by the pair $(i, j) \in \mathcal{E}$. A graph with the property that $(i, j) \in \mathcal{E}$ implies $(j, i) \in \mathcal{E}$ is called undirected. The *neighboring set* N_i of vertex i is defined as $\mathcal{N}_i := \{j \in \mathcal{V} | (i, j) \in \mathcal{E}\}$. For a graph $\mathcal{G}$, its adjacency matrix A is given by $A = (a_{ij})$, with $a_{ii} = 0$ and $a_{ij} = 1$ if $(j, i) \in \mathcal{E}$, and $a_{ij} = 0$ otherwise. The *Laplacian matrix* of $\mathcal{G}$ is defined as $L = (l_{ij})$, where we have $l_{ii} = \sum_{j \neq i} a_{ij}$, $l_{ij} = -a_{ij}$, $i \neq j$. Since all row-sums of L are zero, i.e. $\sum_j l_{ij} = 0\ \forall i$, zero is an eigenvalue of L with eigenvector $\mathbf{1} := (1, \ldots, 1)^T$. Consequently, L has at most rank $p - 1$.

In the case that $\mathcal{G}$ is undirected, the Laplacian L is a positive semi-definite real symmetric matrix. The Laplacian matrix of an undirected graph has rank $p - 1$ if and only if the graph is connected. Under this condition, the zero eigenvalue of L has multiplicity one. The $p - 1$ nonzero eigenvalues of L can be ordered increasingly as $0 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_p$. Furthermore, L can be *diagonalized* by an orthogonal transformation U that brings it to the form $\Lambda := U^TLU = \mathrm{diag}(0, \lambda_2, \ldots, \lambda_p)$, which is denoted by Λ.

For a general directed graph $\mathcal{G}$, L is not necessarily symmetric, and the eigenvalues of L are not guaranteed to be real. In this case, still, all eigenvalues of L have nonnegative real part. A directed graph contains a spanning tree if and only if its Laplacian has rank $p - 1$. In this case, the set of nonzero eigenvalues of L is denoted in arbitrary order by $\{\lambda_2, \lambda_3, \ldots, \lambda_p\}$, and L can be brought to *upper triangular form* by a complex unitary transformation U: $U^*LU = \Lambda_u$, where Λ_u is a complex upper triangular matrix with $0, \lambda_2, \ldots, \lambda_p$ on the diagonal.

4. Multi-agent systems

In this section, the problem of synchronization of multi-agent systems is briefly reviewed. We consider multi-agent systems with p agents, where the communication topology of the system is represented by a directed or undirected graph $\mathcal{G}$ with Laplacian matrix L. For each agent i of the network, the nominal agent dynamics is given by one and the same finite-dimensional linear time-invariant system

$$\dot{x}_i = Ax_i + Bu_i, \qquad y_i = Cx_i. \tag{2}$$

For each i, the state x_i takes its values in $\mathbb{R}^n$, and the input signal u_i and output signal y_i take values in $\mathbb{R}^m$ and $\mathbb{R}^q$, respectively. It is a standing assumption in this paper that (A, B) is stabilizable and (C, A) is detectable.

Following [10,11], these agents are then interconnected using an observer-based *dynamic* protocol of the form

$$\begin{aligned} \dot{w}_i &= Aw_i + B\sum_{j \in \mathcal{N}_i}(u_i - u_j) + G\sum_{j \in \mathcal{N}_i}\Big((y_i - y_j) - Cw_i\Big), \\ u_i &= Fw_i \end{aligned} \tag{3}$$

for $i = 1, 2, \ldots, p$. The structure of this protocol is as follows. Each controller is able to observe the disagreement output signal $\sum_{j \in \mathcal{N}_i}(y_i - y_j)$ and the relative input $\sum_{j \in \mathcal{N}_i}(u_i - u_j)$ of its corresponding agent. The differential equation in (3) acts as an *observer* for the relative state $\sum_{j \in \mathcal{N}_i}(x_i - x_j)$ of agent i. The protocol state w_i is an estimate of this quantity. It is easily verified that the error $e_i := w_i - \sum_{j \in \mathcal{N}_i}(x_i - x_j)$ has error dynamics $\dot{e}_i = (A - GC)e_i$, which is asymptotically stable if $A - GC$ is Hurwitz. This estimate is then fed back to the agent by means of a static feedback.

By interconnecting the agents (2) using the above protocol, we obtain the closed-loop dynamics of the entire network. Denote $\mathbf{x} = \mathrm{col}(x_1, x_2, \ldots, x_p)$, $\mathbf{u} = \mathrm{col}(u_1, u_2, \ldots, u_p)$, $\mathbf{y} = \mathrm{col}(y_1, y_2, \ldots, y_p)$, and $\mathbf{w} = \mathrm{col}(w_1, w_2, \ldots, w_p)$. The network dynamics is now

$$\begin{pmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{w}} \end{pmatrix} = \begin{pmatrix} I \otimes A & I \otimes BF \\ L \otimes GC & I \otimes (A - GC) + L \otimes BF \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{w} \end{pmatrix}. \tag{4}$$

Next, we state the prevalent definition of synchronization of such a network.

Definition 2. The network with agent dynamics (2) is said to be synchronized by protocol (3) if for all $i, j = 1, 2, \ldots, p$ we have that $x_i(t) - x_j(t) \to 0$ and $w_i(t) - w_j(t) \to 0$ as $t \to \infty$.

In [11], it is shown that synchronization of the network with agent dynamics (2) by protocol (3) is equivalent to the stabilization of a single linear system by each controller from a given set of $p - 1$ controllers. We will use a similar argument in the next section, where we examine the synchronizability of multi-agent system in which the agent dynamics is a coprime factor perturbation of the nominal agent dynamics.

5. Robust synchronization

While the nominal agent dynamics is still given by the unperturbed dynamics (2), we now allow uncertainty in the form of coprime factor perturbations of the nominal agent dynamics. In [2], this paradigm for model uncertainty was used in the context of optimal robust stabilization, see also [3]. The agents have identical nominal transfer matrices given by $G(s) = C(sI - A)^{-1}B$. It is well known that there exists a coprime factorization of G of the form $G = M^{-1}N$ with $M, N \in \mathcal{RH}_\infty$ such that $NN^* + MM^* = I$, where $N^*(s) := N^T(-s)$, see e.g. [2,3]. Such a factorization is called a normalized coprime factorization over $\mathcal{RH}_\infty$. Such factorization can be obtained by means of the algebraic Riccati equation

$$AQ + QA^T - QC^TCQ + BB^T = 0. \tag{5}$$

By detectability it has a unique real symmetric solution such that $A - QC^TC$ is Hurwitz. This matrix Q is called the stabilizing solution of (5). A normalized coprime factorization $G = M^{-1}N$ is then obtained by taking $M(s) := I - C(sI - A + QC^TC)^{-1}QC^T$ and $N(s) := C(sI - A + QC^TC)^{-1}B$. In this paper we consider the situation that the transfer matrices of the agents are coprime factor perturbations of the nominal transfer matrix, i.e. the transfer matrix $G = M^{-1}N$ of agent i is perturbed to

$$G_{(\Delta_M^i, \Delta_N^i)} := (M + \Delta_M^i)^{-1}(N + \Delta_N^i),$$