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From local averaging to emergent global behaviors: The fundamental role of network interconnections

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This work is dedicated to the memory of Jan C. Willems

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ABSTRACT

Distributed averaging is one of the simplest and most widely studied network dynamics. Its applications range from cooperative inference in sensor networks, to robot formation, to opinion dynamics. A number of fundamental results and examples scattered through the literature are gathered here and some original approaches and generalizations are presented, emphasizing the deep interplay between the network interconnection structure and the emergent global behavior.

1. Introduction

One of the core concepts in the behavioral approach to systems and control developed by Jan Willems in the '80s is that of interconnection [1]. Encompassing the traditional notion of feedback interconnection on which classical input/output control theory is based, the behavioral approach allows for defining interconnections of systems at a more primitive level, as intersections of solution sets of the evolution equations, without the need for specific flow diagrams. As Jan used to repeat, what is an input and what is an output is a matter of the application. This idea of going beyond the classical input/output formalism proved fruitful in applications, e.g., in coding theory, where Willems' study of minimal state space realizations [2] laid the foundations of trellis representations which are the basic tool for the design of efficient decoding algorithms.

More recently, the study of network dynamics is showing deep cultural analogies with the *ansatz* of the behavioral approach. Network dynamics entail a large number of (relatively) simple systems coupled together along the architecture of a graph. The overall dynamical system can thus be seen as the interconnection of these atomic devices. It does not make much sense to classify *a priori* interconnection signals as input or outputs, rather they are variables coupling the systems, possibly sensor measurements, state positions, epidemic states, and it is often impossible to say who is influenced by whom. The emergence of global behaviors such as synchronization, information fusion, polarization, and diffusion is one of the distinctive features of these complex interconnected systems. Such global behaviors can in fact be seen as the result of the local interactions and of the interconnection graph structure.

This paper focuses on a particularly simple and well studied class of network dynamics: distributed averaging systems [3,4]. These are linear network dynamics exhibiting many interesting collective behaviors, such as synchronization and transition phenomena. Their applications range from inferential sensor network algorithms [5], to network vehicle formation [6], to models for opinion dynamics [7]. Most of the behavioral approach developed by Jan was in fact focused on linear systems: he used to say that linear systems are sufficiently rich from a theoretical viewpoint and yet containing a huge variety of applications. Keeping models as simple as possible was central in Jan's approach to science.

Using classical results from the Perron–Frobenius theory of non-negative matrices, we first present an asymptotic analysis of the linear averaging dynamics on arbitrary interconnection graphs. As expected, the graph topology plays a crucial role in shaping the

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emergent global behavior. While it is well known that all states reach an asymptotic consensus on connected graphs, Theorem 2 of Section 2 analyzes the case of a general graph and shows that the asymptotic state of every agent in the network turns out to be a convex combination of the consensus reached by the sink connected components (i.e., components with no outgoing links). The weights of such convex combination have several useful interpretations. They can be seen as hitting probabilities of the dual Markov chain generated by the same averaging matrix or, when the graph is undirected, as voltages of an electrical circuit with suitable boundary conditions on the nodes belonging to the sink components, as explained in Section 3. While analogous electrical interpretations are well known in Markov chain theory [8,9], they have received relatively minor attention in the distributed averaging literature, with a few exceptions. In particular, to our knowledge, Theorem 3 has not appeared elsewhere in this generality.

A relevant case in the applications is when the sink components all consist of single nodes – called stubborn nodes – that never change their state, e.g., playing the role of opinion leaders in social networks, or anchor nodes in robot formation control. The final part of the paper is dedicated to a deeper understanding of how the asymptotic state is distributed within the network in the presence of such stubborn nodes. It turns out that – depending on the stubborn nodes' centrality and the graph connectivity – quite different phenomena can emerge ranging from polarized to homogeneous equilibrium configurations [10]. In the polarized case, nodes tend to cluster in subfamilies and converge to values very close to that of a particular stubborn agent, whereas in the homogeneous regime most of the nodes tend to get close to a consensus on a value which is a convex combination of the stubborn node values. In Section 4, we present these phenomena through an example where the transition between the two regimes can be analyzed in detail. We then recall more general results appeared in the literature.

We gather here some notational conventions. The transpose of A is denoted by A'; $\mathbb{1}$ is the all-1 vector; $\mathbb{1}_{\mathcal{A}}$ is the vector with all entries equal to 0 except for those whose label is in $\mathcal{A}$ that are equal to 1. The asymptotic notation $a \ll b$ and $a \sim b$ means $\lim a/b = 0$ and $\lim a/b = 1$, respectively.

2. Averaging dynamics on general graphs

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ be a directed weighted graph representing the network, where $\mathcal{V} = \{1, \dots, n\}$ is the set of nodes, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of links, and $W \in \mathbb{R}^{n\times n}$ is a matrix of nonnegative link weights such that $W_{ij} > 0$ if and only if $(i,j) \in \mathcal{E}$, with positive diagonal elements of W corresponding to self-loops. We refer to the graph $\mathcal{G}$ as: *connected* if W is irreducible;[2] *undirected* if W is symmetric; *balanced* if $W\mathbb{1} = W'\mathbb{1}$; *unweighted* if $W_{ij} \in \{0, 1\}$ for all $i, j \in \mathcal{V}$. We denote the out-degree vector by $w = W\mathbb{1}$ and assume[3] that $w_i > 0$ for all nodes i. We then introduce the matrices

$$D = \operatorname{diag}(w), \qquad P = D^{-1}W, \qquad L = D - W. \tag{1}$$

Observe that the matrices P and $-L$ are respectively row-stochastic and Metzler. Also, $P'w = w$ and $L'\mathbb{1} = 0$ if and only if $\mathcal{G}$ is balanced. Moreover, $\mathcal{G}$ being undirected is equivalent to the detailed balance $w_iP_{ij} = w_jP_{ji}$ for $i, j \in \mathcal{V}$, a property that is referred to as reversibility of P (with respect to w). The matrix L is known as the graph *Laplacian*.

One of the most popular network dynamics can be seen as the interconnection of local averaging systems, i.e., multi-input/single-state dynamics placed at the nodes $i \in \mathcal{V}$ and governed by the linear updates $x_i(t+1) = \alpha x_i(t) + (1-\alpha)\sum_j P_{ij}u_j(t)$. Here, $\alpha \in [0, 1]$ is an inertia parameter. By putting $u_j(t) = x_j(t)$ one obtains the interconnected system

$$x_i(t+1) = \alpha x_i(t) + (1-\alpha)\sum_j P_{ij}x_j(t), \tag{2}$$

for $i \in \mathcal{V}$. In (2), the sum index j runs in principle over the whole node set $\mathcal{V}$, but is in fact restricted to the out-neighborhood $\mathcal{N}_i := \{j : W_{ij} > 0\}$ of node i in $\mathcal{G}$. By assembling all the node states in a column vector $x(t) \in \mathbb{R}^n$, (2) can be compactly rewritten as

$$x(t+1) = P_\alpha x(t), \tag{3}$$

where $P_\alpha = \alpha I + (1-\alpha)P$. Hence, the state vector $x(t)$ of the *distributed averaging* dynamics (3) evolves as $x(t) = P_\alpha^t x(0)$, so that its asymptotic behavior is dictated by the eigen-structure of P_α. Being a stochastic matrix, P is non-expansive in the $\|\cdot\|_\infty$ norm, so that its spectrum is contained in the unitary disk centered in 0. Hence, for $0 \le \alpha \le 1$, the matrix P_α has 1 as eigenvalue (corresponding to right eigenvector $\mathbb{1}$) and its whole spectrum is contained in the closed disk of diameter coinciding with the segment joining the points $-1+2\alpha$ and 1 in the complex plane. Finer properties of the spectrum of P_α are closely related to the geometrical properties of the graph $\mathcal{G}$ as summarized below.

First we consider the case when the graph $\mathcal{G}$ is connected. In this case, it is a standard result of the Perron–Frobenius theory that P_α^t converges to a matrix $\mathbb{1}\pi'$ where π can be uniquely characterized as the left eigenvector $\pi' = \pi'P$ such that $\mathbb{1}'\pi = 1$. Connectivity of $\mathcal{G}$ implies that all the entries of π – which is referred to as the *centrality* vector – are strictly positive. For a balanced graph, π is proportional to the degree vector, namely, $\pi = w/(\mathbb{1}'w)$. For general, unbalanced, connected graphs such simple expression does not hold true, while one can express the entries π_i in terms of infinite sums. For $\alpha \in [0, 1)$, let the *mixing time* of P_α be

$$\tau_\alpha := \inf\Big\{t \ge 0 : \max_{i\in\mathcal{V}} \sum_j |(P_\alpha^t)_{ij} - \pi_j| \le \frac{1}{2e}\Big\}.$$

The mixing time is a popular index to study the speed of convergence of P_α^t. In certain cases it can be estimated from knowledge of the second largest eigenvalue of P_α or coupling techniques. E.g., for the unweighted d-dimensional toroidal grid, one has $\tau_\alpha \sim C_d n^{2/d}$ where C_d is a constant depending on the dimension d but not on the graph size n. For general large-scale graphs whose spectrum analysis is unfeasible and for which no effective couplings are known, it proves more convenient to relate the mixing time to the graph *conductance*

$$\Phi := \min_{\emptyset\neq\mathcal{U}\subsetneq\mathcal{V}} \frac{\sum_{i\in\mathcal{U}}\sum_{j\in\mathcal{V}\setminus\mathcal{U}} \pi_i P_{ij}}{\sum_{i\in\mathcal{U}}\pi_i \cdot \sum_{j\in\mathcal{V}\setminus\mathcal{U}}\pi_j},$$

that is a measure of the lack of bottlenecks in the graph. Results in [11, Section 4.3] imply that

$$\frac{1-2/e}{\Phi} \le \tau_{1/2} \le \frac{1}{\Phi^2}\log\frac{e^2}{\pi_*}, \tag{4}$$

where $\pi_* = \min_{i\in\mathcal{V}}\pi_i$. By combining the bounds above with estimates of the conductance, it can be shown, e.g., that the Erdos–Renyi random graphs in the connected regime[4] exhibit, with

[2] Note that this convention deviates from the one adopted by some authors who refer to $\mathcal{G}$ as *strongly* connected if W is irreducible and simply connected if $W + W'$ is irreducible.

[3] This assumption implies no loss of generality since one can add a self-loop with $W_{ii} > 0$ to nodes i with $w_i = 0$ without modifying connectivity and other properties of $\mathcal{G}$.

[4] They are constructed by considering n nodes randomly putting a link between any pair of them independently with probability $p = c\log n/n$ for $c > 1$.