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Control system with hysteresis and delay

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ABSTRACT

A class of control-delay systems exhibiting a hysteresis behavior is considered. Existence of solutions and a relaxation result are obtained for this system.

1. Introduction

Let $r > 0$ be a finite delay. For an interval $T \subset \mathbb{R}$ denote by $C(T, \mathbb{R}^n)$, $n = 1, 2$, the space of continuous functions from T to $\mathbb{R}^n$ equipped with the sup-norm. In particular, $\mathcal{C}_0 := C([-r, 0], \mathbb{R})$ is the space of continuous functions from $[-r, 0]$ to $\mathbb{R}$. The norm on $\mathcal{C}_0$ we will denote by $|\cdot|_\infty$. For $x \in C([-r, 1], \mathbb{R})$ define

$$x_t(\tau) := x(t + \tau), \quad \tau \in [-r, 0].$$

In this paper we consider the control system:

$$a_1\dot{v}(t) + a_2\dot{w}(t) = h_1(v_t, w_t)u^1(t) \quad \text{for a.e. } t \in [0, 1], \tag{1.1}$$

$$\dot{w}(t) + \partial I_{K(v(t))}(w(t)) \ni h_2(v_t, w_t)u^2(t) \quad \text{for a.e. } t \in [0, 1], \tag{1.2}$$

$$v(\tau) = v_0(\tau), \qquad w(\tau) = w_0(\tau), \quad \tau \in [-r, 0], \tag{1.3}$$

subject to the mixed control constraint:

$$u(t) = (u^1(t), u^2(t)) \in U(t, v_t, w_t) \quad \text{for a.e. } t \in [0, 1]. \tag{1.4}$$

Here, a_i, $i = 1, 2$, are given constants. The scalar functions $h_i(\cdot, \cdot)$, $i = 1, 2$, are defined on $\mathcal{C}_0 \times \mathcal{C}_0$. The set $K(v) = [f_*(v), f^*(v)]$ is a possibly degenerate interval with $f_*(v), f^*(v)$ being two nondecreasing functions coinciding out of a fixed interval. The operator $\partial I_{K(v)}$ is the subdifferential in the sense of the convex analysis of the indicator function of $K(v)$. Furthermore, $U : T \times \mathcal{C}_0 \times \mathcal{C}_0 \to 2^{\mathbb{R}^2}$ is a multivalued mapping with compact, in general, nonconvex values and $v_0, w_0 \in \mathcal{C}_0$ are such that

$$w_0(0) \in K(v_0(0)).$$

Recall that the subdifferential of the indicator function $I_{K(v)}$, $v \in \mathbb{R}$,

$$I_{K(v)}(w) := \begin{cases} 0 & \text{if } w \in K(v), \\ +\infty & \text{otherwise,} \end{cases}$$

of the interval $K(v) = [f_*(v), f^*(v)]$ has the form:

$$\partial I_{K(v)}(w) = \begin{cases} \emptyset & \text{if} \quad w \notin [f^*(v), f_*(v)], \\ [0, +\infty) & \text{if} \quad w = f^*(v) > f_*(v), \\ \{0\} & \text{if} \quad f_*(v) < w < f^*(v), \\ (-\infty, 0] & \text{if} \quad w = f_*(v) < f^*(v), \\ (-\infty, +\infty) & \text{if} \quad w = f_*(v) = f^*(v). \end{cases} \tag{1.5}$$

The differential inclusion (1.2) is equivalent to an operator equation with a hysteresis operator of generalized play type with generating curves f_* and f^*. This equation defines a hysteresis relationship between two functions: the input v and the output w. Hysteresis phenomena are encountered in a large variety of real world situations. The areas of investigation where they arise range from thermomechanics to population dynamics and economics. The mathematical treatment of hysteresis has received a considerable attention during the last three decades (cf., e.g., monographs [1–4]). Optimal control problems for systems describing hysteresis effects also appeared in a number of recent works [5–10].

Despite the fact that already a rather simple model exhibiting hysteresis behavior, that of an automotive thermostat, is readily exposed to a delay effect [11], there are literally just a few papers considering differential systems with interplay of hysteresis and delay [12,13]. Moreover, to the best of the author's knowledge, there have been no contributions so far dealing with control

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systems combining the two phenomena. The present paper intends to start the corresponding research.

A physically simplified model of the dynamic behavior of thermostats in cars controlling the operating temperature of the engine considered in [11,12] is given by the following system:

$$\dot\theta(t) = q_e - q_r\omega(t-\tau), \quad t \ge 0, \tag{1.6}$$

$$\dot\omega(t) \in -\partial I_{J(\theta(t))}(\omega(t)), \quad t \ge -\tau, \tag{1.7}$$

$$\theta(t) = \theta_0(t), \qquad -\tau \le t \le 0, \qquad \omega(-\tau) = \omega_0, \tag{1.8}$$

for the unknown temperature of the coolant fluid $\theta(t)$ and the fractional thermostat opening $\omega(t)$. Here, q_e is the engine heat generation, q_r is the cooling power of the radiator assumed to be constant, θ_0 is the initial condition for the temperature over the interval $[-\tau, 0]$, ω_0 is the initial value of the thermostat opening, τ is the delay. The appearance of hysteresis is explained by the difference of the way the thermostat opens when the temperature rises from the way it closes when the temperature falls, and the hysteresis region is described by $J(\theta) = [f_R(\theta), f_L(\theta)]$, $\theta \in \mathbb{R}$, with f_R, f_L being two given curves. The delay τ is induced by the cooling loop, this is the time the cooling flow takes to run from the radiator to the engine.

The Ref. [13] deals with a model similar to that described by (1.6)–(1.8) with a different hysteresis operator to study a physical system consisting of a solid body kept around a certain temperature range by a thermostat in the situation when the temperature of the body is measured at a place on the body different from the place where the heat of the thermostat is applied, thus introducing a delay into the system.

Note that (1.6)–(1.8) is a special case of our evolution system (1.1)–(1.3) with a fixed control u. In particular, a possible optimization of the engine cooling process can be achieved in a natural way by a partial control of the heat exchange in the radiator in the energy balance equation (1.6).

Along with (1.4) we consider the following alternative constraint:

$$u(t) \in \overline{\mathrm{co}}\, U(t, v_t, w_t) \quad \text{a.e. on } [0, 1]. \tag{1.9}$$

where $\overline{\mathrm{co}}$ stands for the closed convex hull.

A *solution* of control system (1.1)–(1.4) is a pair (x, u), $x = (v, w) \in C([-r, 1], \mathbb{R}^2)$, $u = (u_1, u_2) \in L^2(T, \mathbb{R}^2)$ such that the restriction $x|_{[0,1]}$ is absolutely continuous, $v(\tau) = v_0(\tau)$, $w(\tau) = w_0(\tau)$, $\tau \in [-r, 0]$, and (1.1), (1.2), (1.4) hold a.e. on $[0, 1]$. A solution of (1.1)–(1.3), (1.9) is defined similarly.

Note that from (1.2) and (1.5) it follows that for a solution (x, u), $x = (v, w)$, we necessarily have $w|_{[0,1]} \in K(v|_{[0,1]})$.

In the present paper we prove the existence of solutions to system (1.1)–(1.3) with both constraints (1.4) and (1.9). Then, we show that any solution subject to the latter convexified constraint can be approximated by solutions of (1.1)–(1.4). This important in the control theory property is usually referred to as *relaxation*.

2. Preliminaries and assumptions

Let H be a Hilbert space with the inner product $\langle \cdot, \cdot \rangle$. A function $\varphi : H \to \mathbb{R} \cup \{+\infty\}$ is said to be proper if its effective domain

$$\mathrm{dom}\,\varphi = \{x \in H : \varphi(x) < +\infty\}$$

is nonempty. By definition, the subdifferential $\partial\varphi(x)$, $x \in H$, of a proper, convex, lower semicontinuous function φ is the set

$$\partial\varphi(x) = \{h \in H : \langle h, y - x\rangle \le \varphi(y) - \varphi(x),\ \forall y \in H\}.$$

The subdifferential $\partial\varphi : H \to 2^H$ is a monotone operator. Recall that a multivalued operator $A : H \to 2^H$ is called monotone if for any $x, y \in \mathrm{dom}\,A$, $\mathrm{dom}\,A = \{x \in H : Ax \ne \emptyset\}$, and any $h_1 \in Ax$ and $h_2 \in Ay$, the inequality $\langle x - y, h_1 - h_2\rangle \ge 0$ holds.

Let $\|\cdot\|$ denote the norm on the Euclidean space $\mathbb{R}^2$. The Hausdorff metric on the space of nonempty compact subsets from $\mathbb{R}^2$ we denote by $\mathrm{haus}(\cdot,\cdot)$.

Let T be an interval of the real line $\mathbb{R}$. We call a multivalued mapping $F : T \to 2^{\mathbb{R}^2}$ measurable if $\{t \in T : F(t) \cap V \ne \emptyset\}$ belongs to the σ-algebra of Lebesgue measurable subsets of T for any closed set $V \subset \mathbb{R}^2$.

For a Banach space X the notation ω-X means that the space X is equipped with the weak topology. The same notation is used for subsets of X with the topology induced by that of the space ω-X.

We make the following assumptions on the functions describing our system (1.1)–(1.4):

Hypothesis H(f). The functions f_*, f^* defining the hysteresis region $K(v)$ in (1.2) are such that $f_*(v) \le f^*(v)$, $v \in \mathbb{R}$, and

(1) f_*, f^* are nondecreasing and Lipschitz continuous on $\mathbb{R}$;
(2) there exists $k_0 > 0$ such that $f_*(v) = f^*(v)$ for $v \in (-\infty, -k_0] \cup [k_0, +\infty)$.

Hypothesis H(a). The constants $a_i > 0$, $i = 1, 2$;

Hypothesis H(h). The functions $h_i : \mathcal{C}_0 \times \mathcal{C}_0 \to \mathbb{R}$, $i = 1, 2$, have the properties:

(1) there exists $C_i > 0$, $i = 1, 2$, such that

$$|h_i(v, w)| \le C_1 + C_2\|(v, w)\|_\infty, \quad v, w \in \mathcal{C}_0, \tag{2.1}$$

where $\|\cdot\|_\infty$ is the sup-norm on the space $C([-r, 0], \mathbb{R}^2)$;

(2) h_i, $i = 1, 2$, are Lipschitz continuous on $\mathcal{C}_0 \times \mathcal{C}_0$:

$$|h_i(v_1, w_1) - h_i(v_2, w_2)| \le L(|v_1 - v_2|_\infty + |w_1 - w_2|_\infty) \tag{2.2}$$

for some $L > 0$, and any $v_i, w_i \in \mathcal{C}_0$, $i = 1, 2$.

Hypothesis H(U). The multivalued mapping $U : T \times \mathcal{C}_0 \times \mathcal{C}_0 \to 2^{\mathbb{R}^2}$ admits compact values and is such that:

(1) the mapping $t \to U(t, v, w)$, $v, w \in \mathcal{C}_0$, is measurable;
(2) there exists a function $k(\cdot) \in L^2(T, \mathbb{R}^+)$ such that

$$\mathrm{haus}(U(t, v_1, w_1), U(t, v_2, w_2)) \le k(t)\, \|(v_1, w_1) - (v_2, w_2)\|_\infty \tag{2.3}$$

for any $v_i, w_i \in \mathcal{C}_0$, $i = 1, 2$, a.e. on $[0, 1]$;

(3) the following inequality holds:

$$\|U(t, v, w)\| = \sup\left\{\left\|\left(u^1, u^2\right)\right\|;\ \left(u^1, u^2\right) \in U(t, v, w)\right\} \le m \tag{2.4}$$

for a.e. $t \in [0, 1]$, $v, w \in \mathcal{C}_0$, and a positive constant m.

3. Existence and uniqueness for a fixed control *u*

In this section we prove that system (1.1)–(1.3) with a fixed control $u \in L^2([0, 1], \mathbb{R}^2)$ has a unique solution.

Consider the system:

$$a_1\dot v(t) + a_2\dot w(t) = \varphi^1(t) \quad \text{a.e. on } [0, 1], \tag{3.1}$$

$$\dot w(t) + \partial I_{K(v(t))}(w(t)) \ni \varphi^2(t) \quad \text{a.e. on } [0, 1], \tag{3.2}$$

$$v(0) = v_0(0), \qquad w(0) = w_0(0), \tag{3.3}$$

for $\varphi^1, \varphi^2 \in L^2([0, 1], \mathbb{R})$. Its *solution* is a pair $(v, w) \in W^{1,2}([0, 1], \mathbb{R}^2)$, $v(0) = v_0(0)$, $w(0) = w_0(0)$, $w \in K(v)$, such that (3.1), (3.2) hold.