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# On stabilizing linear systems with input saturation via soft variable structure control laws

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ABSTRACT

A large number of control methods deal with stabilizing single-input linear systems with input saturation. The linear full-state feedback is one of the most used, yielding either nonsaturating or high gain controls, whereby the saturating effects of the latter are reduced by means of anti-windup structures. Among the nonlinear controls, the time optimal control, in this case a bang–bang type control, yields the fastest time response, but its switching surface is generally not characterizable. Related to the speed of the time response is the convergence rate, which can be determined using invariant sets. The most used ones are the ellipsoidal sets, since they can be analyzed using powerful tools such as the Lyapunov equation and designed via convex optimization. For this reason, they are also used for designing soft variable structure controls. The paper presents a nonconservative design method for a stabilizing control of this type employing implicit Lyapunov functions (iLF). A nonsaturating control law is given, including some infinitely densely nested and contractive invariant sets of the equilibrium state. The control law is then optimized by maximizing the iLF-based lower bound of the convergence rate. The maximal convergence control is shown to be of bang–bang type, with a parameter dependent switching scheme. To overcome possible difficulties of a switching controller, a saturating high gain control is also presented.


## 1. Introduction

For LTI systems with input saturation, the problem of achieving a fast response can be solved using the time optimal control, which, in this case, is a bang–bang type control. However, since it is generally not possible to characterize the switching surface, the time optimal control law is usually given only for systems with low dimensions. Also, the discontinuity of the bang–bang control law may cause technical difficulties, such as the uninterrupted activity of the control due to unavoidable noise, which is unnecessary and may damage the actuators.

A fast response may also be related to the convergence rate of a stable system, which is defined as the smallest decay factor by which the norm of the state trajectory converges to the equilibrium state. For linear systems, the convergence rate is the same for the entire state space and corresponds to the absolute value of the real part of the pole that is closest to the imaginary axis. For nonlinear systems it depends on the distance to the equilibrium state. For this purpose, the theory of invariant sets has been used to guarantee a lower bound of the convergence rate inside an invariant set, where also the range of the systems' input is not exceeded. Different types of sets have been used, mainly ellipsoidal sets, see for example [1], and polyhedral sets, see for example [2]. Using Lyapunov functions to describe the invariant sets, a lower bound of the convergence rate is given by the minimal decay factor of the Lyapunov function along the trajectories of the system that start inside the invariant set. The maximal convergence control for LTI systems with input saturation can be shown to be of bang–bang type, having a simple switching scheme that involves the given Lyapunov function. Design procedures can be found for example in [3,4].

Another method to increase the speed of the time response is by employing variable structure controls (VSC), with or without sliding modes. The latter, which intentionally preclude sliding modes, continuously vary the controllers' parameters (or structures) according to the distance to the equilibrium state. The first systematically developed method of this type was presented in [5]. A survey of different types of VSCs (without sliding modes) can be found in [6]. One of them is the soft VSC employing implicit Lyapunov-functions (iLFs), which was presented in [7,8]. The parameter variation evolves continuously, and the parameter

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serves as an implicit Lyapunov-function of the system, which guarantees its stability. See for example [9–11] and the references therein for contributions to the stabilization of single-input linear systems by means of this (or a similar) iLF method. Design procedures for soft VSCs using ellipsoidal sets can be found for example in [10,12,13]. They use linear matrix inequalities (LMIs) to describe the stability conditions. However, they are only sufficient, and may render conservative control laws.

The paper presents the necessary and sufficient conditions for the existence of a stabilizing soft VSC employing iLFs with ellipsoidal level sets for single-input LTI systems with input saturation. If the conditions are fulfilled, a bounded control law is given together with some infinitely densely nested and contractive invariant sets of the equilibrium state. The control law is then optimized by maximizing the iLF-based lower bound of the convergence rate. The maximal convergence control is shown to be simply a bang–bang type control with a parameter varying switching scheme. To overcome potential difficulties with the switching controller, the paper presents also a saturating high gain soft VSC employing iLFs, which, at the cost of a slight reduction in the convergence rate, achieves a fast response without the disadvantages of a discontinuous control.

Since the existence conditions are both necessary and sufficient, the presented synthesis method is nonconservative. Furthermore, all existence conditions can be formulated by means of linear matrix inequalities (LMIs), yielding a convex optimization problem. The stabilizing control law possesses also some (constant) parameters, which – under some certain bounds – influence the given contractive invariant set of the equilibrium state and the speed of the time response. The maximal convergence control, which is a bang–bang type control, can be employed to achieve a faster response. In addition, a sufficient gain for its continuous approximation is given in a very general form, depending on the null space of the switching vector.

The paper is organized as follows. Section 2 gives some preliminary notes. Section 3 presents the necessary and sufficient conditions for the existence of a soft VSC by means of iLFs. Section 4 presents the maximal convergence control and the saturating high gain soft VSC employing iLFs. Section 5 shows the design steps of the proposed control laws. Section 6 compares the proposed control laws with the time optimal control by means of an example. Section 7 concludes.

## 2. Preliminary notes

Throughout the paper, the symbols $\exists$ and $\forall$ denote *it exists*, respectively *for all*, $\otimes$ denotes the Kronecker product, and $\|\cdot\|$ denotes the Euclidean norm. $\mathbb{S}^n$ denotes the set of all $n \times n$ symmetric matrices, $\mathbb{P}^n$ denotes the set of all positive definite matrices, and $\mathbf{A} \succ \mathbf{B}$ means that $\mathbf{A} - \mathbf{B} \in \mathbb{P}^n$. Furthermore, $\mathcal{N}(\mathbf{A})$ denotes the null space of a matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$, i.e. $\mathcal{N}(\mathbf{A}) := \{\mathbf{x} \in \mathbb{R}^n | \mathbf{A}\mathbf{x} = \mathbf{0}\}$, and $\mathcal{L}(u, \beta)$ denotes a linear region of saturation, i.e. $\mathcal{L}(u, \beta) := \{\mathbf{x} \in \mathbb{R}^n | \, |u(\mathbf{x})| \le \beta\}$. For parameter dependent matrices and vectors the following notations are used: $\mathbf{X}_v := \mathbf{X}(v)$ and $\mathbf{x}_v := \mathbf{x}(v)$. Finally, $\mathcal{G}(v)$ denotes an ellipsoidal region, i.e. $\mathcal{G}(v) := \{\mathbf{x} \in \mathbb{R}^n | \mathbf{x}^\top \mathbf{P}_v \mathbf{x} < 1\}$, with $\mathbf{P}_v \succ \mathbf{0}$. Also, the following definitions will be used:

**Definition 2.1** (*Contractive Invariant Set*)**.** Consider the dynamic system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^n$, having an equilibrium state $\mathbf{x}_R = \mathbf{0}$, and a unique solution for every initial state. For a given matrix $\mathbf{P}_v \succ \mathbf{0}$, the closed set $\mathcal{G}(v) := \{\mathbf{x} \in \mathbb{R}^n | \mathbf{x}^\top \mathbf{P}_v \mathbf{x} < 1\}$ is called *contractive invariant* if

$$\frac{\partial \mathbf{x}^\top \mathbf{P}_v \mathbf{x}}{\partial \mathbf{x}} \cdot \dot{\mathbf{x}} = 2\mathbf{x}^\top \mathbf{P}_v \cdot \mathbf{f}(\mathbf{x}) < 0, \quad \forall \mathbf{x} \in \mathcal{G}(v) \setminus \{\mathbf{0}\}.$$

It is furthermore positively invariant, since all trajectories that start within the set, remain in it for all future time.

**Definition 2.2** (*Nested Ellipsoidal Regions*)**.** The sets $\mathcal{G}(v_1)$ and $\mathcal{G}(v_2)$, with $0 < v_2 < v_1$, are called *nested*, if their boundaries, $\partial\mathcal{G}(v_1)$ and $\partial\mathcal{G}(v_2)$, have no common points, that is $\partial\mathcal{G}(v_1) \cap \partial\mathcal{G}(v_2) = \emptyset$, and $\mathcal{G}(v_2) \subset \mathcal{G}(v_1)$.

**Definition 2.3** (*Full-Rank Factorization*)**.** Let $\mathbf{A} \in \mathbb{R}^{m \times n}$, with $\text{rank}(\mathbf{A}) = r$. The tuple $(\mathbf{A}_l, \mathbf{A}_r)$, where $\mathbf{A}_l \in \mathbb{R}^{m \times r}$, with $\text{rank}(\mathbf{A}_l) = r$, and $\mathbf{A}_r \in \mathbb{R}^{r \times n}$, with $\text{rank}(\mathbf{A}_r) = r$, is called *full-rank factorization* of matrix $\mathbf{A}$ if $\mathbf{A} = \mathbf{A}_l\mathbf{A}_r$.

**Definition 2.4** (*Convergence Rate of an Exponentially Stable Nonlinear System*)**.** The *convergence rate* of an exponentially stable nonlinear system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^n$, is defined to be the maximal decay factor $\alpha > 0$, for which a scalar $\gamma > 0$ exists, so that $\|\mathbf{x}\| \le \gamma \|\mathbf{x}_0\| e^{-\alpha t}, \ \forall t > 0$.

Finally, the following lemma, which is a direct result from [14, Proposition 11.9.5], will be also used:

**Lemma 2.1.** *The following statements are equivalent:*

(i) *The equilibrium state* $\mathbf{x} = \mathbf{0}$ *of the system* $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}, \ \mathbf{x} \in \mathbb{R}^n, \ \mathbf{A} \in \mathbb{R}^{n \times n}$, *is asymptotically stable.*
(ii) $\exists\, \mathbf{P} \in \mathbb{P}^n$, *such that* $\mathbf{A}\mathbf{P} + \mathbf{P}\mathbf{A}^\top \prec \mathbf{0}$.

## 3. Existence conditions for a soft VSC employing iLFs

In the following, we consider single input LTI systems in control normal form, i.e.

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u, \quad \mathbf{x} \in \mathbb{R}^n, \ u \in \mathbb{R}, \ |u| \le 1. \tag{1}$$

The general form of the soft VSC analyzed here is

$$u = -f(\mathbf{x}, v), \tag{2}$$

where, for a given $\mathbf{x}$, the parameter $v \in (0, 1]$ is determined by a selection strategy of the form

$$g(\mathbf{x}, v) = \mathbf{x}^\top \mathbf{P}_v \mathbf{x} - 1 = 0, \tag{3}$$

which divides the state space into ellipsoids, each denoted by $\partial\mathcal{G}(v) := \{\mathbf{x} \in \mathbb{R}^n | \mathbf{x}^\top \mathbf{P}_v \mathbf{x} = 1\}$.

The following theorem, which is a direct result of the theorem given in [8, Theorem 4], presents some sufficient conditions for contractive invariance and nesting of some parameter dependent sets, ensuring also the asymptotic stability of the equilibrium state of the closed-loop system.

**Theorem 3.1** (*Based on [8, Theorem 4]*)**.** *Let* $\mathbf{h}(\mathbf{x})$ *be a continuous function and for* $t \ge 0$, *consider the dynamical system*

$$\dot{\mathbf{x}} = \mathbf{h}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^n,$$

*with the equilibrium state* $\mathbf{x}_R = \mathbf{0}$ *and unique solution for every initial state. If there exists a continuous and differentiable function* $g(\mathbf{x}, v) : \mathcal{H}_0 \to \mathbb{R}$, *with* $\mathcal{H}_0 = \{(\mathbf{x}, v) | \mathbf{x} \in \mathcal{U}_0 \setminus \{\mathbf{0}\}, \ 0 < v \le 1\}$ *and* $\mathcal{U}_0$ *a neighborhood of the origin, that fulfills the conditions*

(S1) *from* $g(\mathbf{x}, v) = 0$, *it follows:* $\mathbf{x} = \mathbf{0} \Leftrightarrow v \to 0^+$,

(S2) $\lim_{v \to 0^+} g(\mathbf{x}, v) > 0$ *and*

$$\lim_{v \to 1^-} g(\mathbf{x}, v) < 0, \quad \forall \mathbf{x} \in \mathcal{U}_0 \setminus \{\mathbf{0}\},$$

(S3) $-\infty < \dfrac{\partial g(v, \mathbf{x})}{\partial v} < 0, \quad \forall (v, \mathbf{x}) \in \mathcal{H}_0,$

(S4) $-\infty < \dfrac{\partial g(v, \mathbf{x}(t))}{\partial t} < 0, \quad \forall (v, \mathbf{x}) \in \mathcal{H}_0,$

*then the equilibrium state* $\mathbf{x} = \mathbf{0}$ *will be asymptotically stable, and the sets*

$$\mathcal{G}(v) := \left\{\mathbf{x} \in \mathbb{R}^n | \, g(\mathbf{x}, v) < 0\right\} \subseteq \mathcal{U}_0$$

*nested and contractive invariant for all* $v \in (0, 1]$.