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A version of a theorem of R. Datko for stability in average

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ABSTRACT

In this note we obtain a version of the well-known theorem of R. Datko for the notion of the exponential stability in average. We consider both cocycles over flows as well as cocycles over maps.
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1. Introduction

In the process of extending the Lyapunov operator equation to the case of autonomous systems $x' = Ax$ when the operator A is unbounded, Datko [1] established his famous theorem which asserts that the trajectories of a C_0-semigroup $\{T(t)\}_{t\geq 0}$ on a Hilbert space X exhibit an exponential decay if and only if they stay in $L^2(\mathbb{R}_+, X)$. Since then this theorem became one of the pillars of the modern control theory and has inspired numerous extensions and generalizations. In particular, Pazy [2] proved that the conclusion of Datko's theorem holds if $L^2(\mathbb{R}_+, X)$ is replaced with any $L^p(\mathbb{R}_+, X)$ with $p \in [1, \infty)$. Furthermore, Datko [3] obtained the version of his theorem which deals with the exponential stability of evolution families $\{T(t, s)\}_{t\geq s\geq 0}$ which describe solutions of the variety of differential equations. More precisely, he proved the following result.

Theorem 1. *Let $\{T(t, s)\}_{t\geq s\geq 0}$ be an evolution family on a Banach space X. The following statements are equivalent:*

(1) *there exist $D, \lambda > 0$ such that*

$$\|T(t, s)\| \leq De^{-\lambda(t-s)} \quad \text{for } t \geq s \geq 0;$$

(2) *there exists $p \in [1, \infty)$ such that*

$$\sup_{s\geq 0} \int_s^\infty \|T(t, s)x\|^p < \infty \quad \text{for each } x \in X.$$

The first results related to discrete-time evolution families are due to Zabczyk [4].

A major improvement of this ideas is due to Rolewicz [5] who characterized exponential stability of evolution families in terms of the existence of appropriate functions N of two real variables (see [6] for details and further discussion). This approach unified and extended many of the previously known results. The most recent contributions [6,7] deal with obtaining the version of Datko's theorem for the notion of nonuniform exponential stability which was introduced by Barreira and Valls (see [8]). Moreover, in [9] the authors have obtained a certain ergodic version of Datko's theorem.

The main purpose of the present paper is to obtain a version of Datko's theorem for the notion of an exponential stability in average which is a particular case of a more general notion of an exponential dichotomy in average introduced in [10,11] for discrete and continuous time respectively. This notion essentially corresponds to assuming the existence of uniform contraction and uniform expansion along complementary directions but now in average, with respect to a given probability measure. We emphasize that this notion includes the classical concepts of uniform exponential dichotomy (and thus also of uniform exponential stability) as particular cases.

The paper is organized as follows. In Section 2 we recall some basic notions and the concept of an exponential stability in average. In Section 3 we prove the version of Datko's theorem for cocycles over semiflows. Then, in Section 4 we do the same but for cocycles over maps. Finally, in Section 5 we imply those results to the study of the persistence of the notion of the exponential stability in average under small linear perturbations.

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2. Preliminaries

We begin by recalling some well-known notions. Let $\Omega = (\Omega, \mathcal{B}, \mu)$ be a probability space. A measurable map $\varphi \colon \mathbb{R}_0^+ \times \Omega \to \Omega$ is said to be a *semiflow* on Ω if:

(1) $\varphi(0, \omega) = \omega$ for $\omega \in \Omega$;
(2) $\varphi(t+s, \omega) = \varphi(t, \varphi(s, \omega))$ for $t, s \geq 0$ and $\omega \in \Omega$.

For each $t \geq 0$ we can consider the map $\varphi_t \colon \Omega \to \Omega$ given by $\varphi_t(x) = \varphi(t, x)$, $x \in \Omega$. Moreover, let X be a Banach space and let $L(X)$ denote the set of all invertible bounded linear operators acting on X. A strongly measurable map $\Phi \colon \mathbb{R}_0^+ \times \Omega \to L(X)$ (this means that $(t, \omega) \mapsto \Phi(t, \omega)x$ is Bochner measurable for each $x \in X$) is said to be a *cocycle* over φ if:

(1) $\Phi(0, \omega) = \text{Id}$ for $\omega \in \Omega$;
(2) $\Phi(t+s, \omega) = \Phi(t, \varphi_s(\omega))\Phi(s, \omega)$ for $t, s \geq 0$ and $\omega \in \Omega$.

Example 1. In the particular case when the map $t \mapsto \Phi(t, \omega)x$ is of class C^1 for each ω and x the cocycle can be described as follows. Let

$$A(\omega) = \frac{d}{dt}\Phi(t, \omega)\Big|_{t=0}.$$

One can easily verify that the unique solution of the problem

$$x' = A(\varphi_t(\omega))x, \qquad x(0) = x_0$$

is then given by $x(t) = \Phi(t, \omega)x_0$. Note that under the above assumption the map $t \mapsto A(\varphi_t(\omega))x$ is continuous for each ω and x.

Before proceeding, we emphasize that cocycles (over maps and flows) arise naturally in the study of nonautonomous dynamics. For example, smooth ergodic theory builds around the study of the derivative cocycle associated either to map or a flow (see Sections 5 and 6 in [12]). Moreover, cocycles describe solutions of variational equations and Cauchy problems with unbounded coefficients (we refer to Chapter 6 of [13] for detailed discussion). Finally, the notion of a cocycle arises from stochastic differential equations (see Chapter 2 in [14] for details).

Let $\mathcal{F}$ denote the Banach space of all Bochner measurable functions, sometimes simply referred to as measurable functions, $z \colon \Omega \to X$ such that

$$\|z\|_1 := \int_\Omega \|z(\omega)\| \, d\mu(\omega) < \infty,$$

identified if they are equal to μ-almost everywhere (we note that $\mathcal{F}$ is simply the set of all Bochner integrable functions identified if they are equal to μ-almost everywhere, sometimes denoted by $\mathcal{L}^1_\mu(\Omega, X)$). Given a cocycle Φ over a semiflow φ, we shall always assume that there exist $K, a > 0$ such that

$$\int_\Omega \|\Phi_\omega(t, \tau)z(\omega)\| \, d\mu(\omega) \leq Ke^{a|t-\tau|} \int_\Omega \|z(\omega)\| \, d\mu(\omega) \tag{1}$$

for $z \in \mathcal{F}$ and $t, \tau \geq 0$, where

$$\Phi_\omega(t, s) = \Phi(t, \omega)\Phi(s, \omega)^{-1}.$$

We now introduce the concept of exponential stability in average. We say that the cocycle Φ is *exponentially stable in average* if there exists $D, \lambda > 0$ such that

$$\int_\Omega \|\Phi_\omega(t, s)z(\omega)\| \, d\mu(\omega) \leq De^{-\lambda(t-s)} \int_\Omega \|z(\omega)\| \, d\mu(\omega), \tag{2}$$

for $z \in \mathcal{F}$ and $t \geq s \geq 0$. This notion is a particular case of a more general notion of exponential dichotomy in mean introduced in [11]. We recall that a cocycle Φ is said to admit an *exponential dichotomy in average* if there exist projections $P_\tau \colon \mathcal{F} \to \mathcal{F}$ for $\tau \geq 0$ such that:

(1) for each $t, \tau \geq 0$ and $z, \bar{z} \in \mathcal{F}$ such that $\bar{z}(\omega) = \Phi_\omega(t, \tau)z(\omega)$ for μ-almost every $\omega \in \Omega$, we have

$$(P_t\bar{z})(\omega) = \Phi_\omega(t, \tau)(P_\tau z)(\omega) \tag{3}$$

for μ-almost every $\omega \in \Omega$;

(2) there exist constants $D, \lambda > 0$ such that for each $z \in \mathcal{F}$, we have

$$\int_\Omega \|\Phi_\omega(t, s)(P_s z)(\omega)\| \, d\mu(\omega) \leq De^{-\lambda(t-s)} \int_\Omega \|z(\omega)\| \, d\mu(\omega) \tag{4}$$

for $t \geq s$ and

$$\int_\Omega \|\Phi_\omega(t, s)(Q_s z)(\omega)\| \, d\mu(\omega) \leq De^{\lambda(t-s)} \int_\Omega \|z(\omega)\| \, d\mu(\omega) \tag{5}$$

for $t \leq s$, where $Q_s = \text{Id} - P_s$.

We note that when $P_t = \text{Id}$, the condition (4) reduces to (2) (while (3) and (5) became trivial) and we recover the notion of exponential stability in average.

Example 2. Any uniformly hyperbolic cocycle admits an exponential dichotomy in average. We recall that a cocycle Φ is *uniformly hyperbolic* if there exist projections $\tilde{P}_t \colon X \to X$ for $t \in \mathbb{R}$ such that:

(1) for each $t, \tau \geq 0$ and $\omega \in \Omega$, we have

$$P_t\Phi_\omega(t, \tau) = \Phi_\omega(t, \tau)P_\tau;$$

(2) there exist constants $D, \lambda > 0$ such that for each $\omega \in \Omega$, we have

$$\|\Phi_\omega(t, \tau)\tilde{P}_\tau\| \leq De^{-\lambda(t-\tau)}$$

for $t \geq \tau$ and

$$\|\Phi_\omega(t, \tau)\tilde{Q}_\tau\| \leq De^{\lambda(t-\tau)}$$

for $t \leq \tau$, where $\tilde{Q}_t = \text{Id} - \tilde{P}_t$.

Defining projections $P_t \colon \mathcal{F} \to \mathcal{F}$ for $t \in \mathbb{R}$ by

$$(P_t z)(\omega) = \tilde{P}_t(z(\omega)),$$

we find that each uniformly hyperbolic cocycle admits an exponential dichotomy in average with respect to any probability measure μ on Ω.

The previous example shows that the notion of an exponential dichotomy in average includes the classical notion of uniform hyperbolicity as a particular case.

Example 3. Now we describe examples of cocycles that admit an exponential dichotomy in average but that are not uniformly hyperbolic. Consider a partition $\Omega = \bigcup_{i=0}^N \Omega_i$ of Ω (N may be finite or infinite) with $\mu(\Omega_0) = 0$ and numbers $\lambda_0 = 0$ and $\lambda_i > 0$ for $i \in \mathbb{N}$ with $\inf_{i\in\mathbb{N}} \lambda_i > 0$. We assume that

$$\int_{\Omega_i} \|\Phi_\omega(t, s)(P_s z)(\omega)\| \, d\mu(\omega) \leq De^{-\lambda_i(t-s)} \int_{\Omega_i} \|z(\omega)\| \, d\mu(\omega)$$

for $t \geq s$ and

$$\int_{\Omega_i} \|\Phi_\omega(t, s)(Q_s z)(\omega)\| \, d\mu(\omega) \leq De^{\lambda_i(t-s)} \int_{\Omega_i} \|z(\omega)\| \, d\mu(\omega)$$

for $t \leq s$, for all $z \in \mathcal{F}$ and $i \in \mathbb{N}_0 \cap [0, N]$. Then the cocycle admits an exponential dichotomy in average. If the set Ω_0 is nonempty, then the cocycle is not uniformly hyperbolic. For example, the set

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