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Stability analysis for planar discrete-time linear switching systems via bounding joint spectral radius

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ABSTRACT

A pair of 2-by-2 matrices can be transformed into the joint forms, which are bound together via an invariant that measures the deformation of their structures with respect to each other. With this, we obtain an approximation of the joint spectral radius of such a pair of matrices from above and then apply it to assessing stability for planar discrete-time linear switching systems.

1. Introduction

Consider the planar discrete-time linear switching system described as follows

$$x(k+1) = A_{\sigma(k)}x(k), \quad x \in \mathbf{R}^2, \tag{1}$$

where σ is the switching function assumed in $\{1, 2\}$ to decide which one of the two subsystems to be switched into activity at each time. System (1) is said to be globally uniformly asymptotically stable if $\lim_{k\to\infty} |x(k; x_0)| = 0$ for any initial state x_0 and any switching function.

The stability problem for planar linear switching systems in continuous-time is the easiest and thus the most understood issue in the research area of switching systems. One of the fruitful results is the characterization on the most unstable trajectory yielded by switching, which is defined in the sense that, at each point on it, it gets away from the origin as far as possible; see, e.g., [1]. Whereas, the analogy in discrete-time remains challenging. From the perspective of optimal control theory, the reason lies in that it fails to characterize the most unstable trajectory in such a way that it has successfully done in the continuous-time case. Yet the joint spectral radius of a finite family of matrices provides us with a framework to look at the problem, which is defined by

$$\hat{\rho}(A_1, A_2) = \lim_{k\to\infty} \sup_{\sigma} \|A_{\sigma(k-1)} \cdots A_{\sigma(1)}A_{\sigma(0)}\|^{\frac{1}{k}}, \tag{2}$$

where $A_{\sigma(k-1)} \cdots A_{\sigma(1)}A_{\sigma(0)}$ is the transition matrix of system (1) for a given switching path σ and the maximization is taken over all possible switching paths. The maximal Lyapunov exponent of system (1) turns out to be the logarithm of the joint spectral radius of $\{A_1, A_2\}$; see [2–4]. Namely,

$$\sup_{x_0\neq 0} \limsup_{k\to\infty} \frac{\ln |x(k; x_0)|}{k} = \ln \hat{\rho}(A_1, A_2).$$

Then, system (1) is stable if and only if $\hat{\rho}(A_1, A_2) < 1$.

Apart from the stability problem concerned here, joint spectral radius is central to a variety of applications; see [5]. However, computing joint spectral radius is an NP-hard problem. Naturally, it has attracted great efforts to characterize the joint spectral radius theoretically or to approximate its truth value at the expense of computational complexity. In particular, its equivalent descriptions can be found in [6] and the references therein. It is worth mentioning that a second-order variational criterion has been derived in [7], which allows to follow such a switching path step by step that maximizes the right-hand side of (2). And the maximization can be achieved because adjusting the switching path in discrete-time can make the spectral radius of the resultant transition matrix continuously vary; see [8].

On the other hand, some methods to study the joint spectral radius are geometric in nature, to which the constructions of special vector norms and their unit balls are central; see, e.g., [9,10].

The aim of this paper is to present a practical upper bound for $\hat{\rho}(A_1, A_2)$, which remains invariant for changing coordinates, and then use it to assess stability. Such a bound indeed has various potential applications, but obtaining it in general is challenging;

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see [11,12]. Our results are derived mainly based on the joint forms of A_1 and A_2, which enable us to capture the deformation of their structures with respect to each other.

Notation: We shall use the following notations. Let $\|X\|$, $\det(X)$, $\mathrm{tr}(X)$, $\rho(X)$ be the 2-norm, determinant, trace, and spectral radius of matrix X, respectively. Then, $\Delta_X = \mathrm{tr}(X)^2 - 4\det(X)$ is the discriminant of the characteristic polynomial of a 2-by-2 matrix X. Let $[X, Y]$ be the Lie commutator of X and Y. Besides, we write j and $\lfloor s \rfloor$ for the imaginary unit and the greatest integer no greater than s, respectively.

2. Main results

Without loss of generality, throughout the paper we assume $\rho(A_1) \geq \rho(A_2)$ and $\Delta_{A_1}\Delta_{A_2} \neq 0$. We now introduce the following invariant

$$\mathcal{K}_{A_1A_2} = 2\frac{\mathrm{tr}(A_1A_2) - \frac{1}{2}\mathrm{tr}(A_1)\mathrm{tr}(A_2)}{\sqrt{|\Delta_{A_1}\Delta_{A_2}|}}, \tag{3}$$

which contains important information about A_1 and A_2; see [13,14]. A basic fact about $\mathcal{K}_{A_1A_2}$ is the following.

Lemma 1 (*[14]*).

$$\det([A_1, A_2]) = \begin{cases} \frac{1}{4}\left(1 - \mathcal{K}^2_{A_1A_2}\right)\Delta_{A_1}\Delta_{A_2}, & \Delta_{A_1}\Delta_{A_2} > 0 \\ \frac{1}{4}\left(1 + \mathcal{K}^2_{A_1A_2}\right)\Delta_{A_1}\Delta_{A_2}, & \Delta_{A_1}\Delta_{A_2} < 0. \end{cases}$$

In particular, $\det([A_1, A_2]) > 0$ *implies* $|\mathcal{K}_{A_1A_2}| < 1$ *and* $\Delta_{A_1}, \Delta_{A_2} > 0$, *and* $\Delta_{A_1}, \Delta_{A_2} < 0$ *implies* $|\mathcal{K}_{A_1A_2}| \geq 1$.

With the construction of $\mathcal{K}_{A_1A_2}$, Balde, Boscain, and Mason proved the following results, which are key for our study.

Lemma 2 (*[15]*). *If* $\det([A_1, A_2]) > 0$, *then there is a linear change of coordinates, which converts* A_1 *and* A_2 *into the following joint forms*

$$\frac{1}{2}\begin{bmatrix} \mathrm{tr}(A_1) & \sqrt{\Delta_{A_1}} \\ \sqrt{\Delta_{A_1}} & \mathrm{tr}(A_1) \end{bmatrix}, \tag{4}$$

$$\frac{1}{2}\begin{bmatrix} \mathrm{tr}(A_2) + \hat{\mathcal{K}}_{A_1A_2}\sqrt{\Delta_{A_2}} & \mathcal{K}_{A_1A_2}\sqrt{\Delta_{A_2}} \\ \mathcal{K}_{A_1A_2}\sqrt{\Delta_{A_2}} & \mathrm{tr}(A_2) - \hat{\mathcal{K}}_{A_1A_2}\sqrt{\Delta_{A_2}} \end{bmatrix}, \tag{5}$$

where $\hat{\mathcal{K}}_{A_1A_2} = \sqrt{1 - \mathcal{K}^2_{A_1A_2}}$.

Lemma 3 (*[15]*). *If* $\det([A_1, A_2]) < 0$, *then there is a linear change of coordinates, which converts* A_1 *and* A_2 *into the following joint forms*

$$\frac{1}{2}\begin{bmatrix} \mathrm{tr}(A_1) & \sqrt{|\Delta_{A_1}|} \\ \mathrm{sgn}(\Delta_{A_1})\sqrt{|\Delta_{A_1}|} & \mathrm{tr}(A_1) \end{bmatrix}, \tag{6}$$

$$\frac{1}{2}\begin{bmatrix} \mathrm{tr}(A_2) & \frac{\mathrm{sgn}(\Delta_{A_2})}{F}\sqrt{|\Delta_{A_2}|} \\ F\sqrt{|\Delta_{A_2}|} & \mathrm{tr}(A_2) \end{bmatrix}, \tag{7}$$

where $F \in \mathbf{R}$ *satisfies* $|F| \geq 1$ *and*

$$F + \frac{\mathrm{sgn}(\Delta_{A_1}\Delta_{A_2})}{F} = 2\mathcal{K}_{A_1A_2}. \tag{8}$$

Since joint spectral radius is independent of a particular choice of coordinates, hereafter we tacitly suppose that A_1 and A_2 are in the joint forms of (4) and (5) (resp., (6) and (7)) when $\det([A_1, A_2]) > 0$ (resp., < 0).

Let

$$\Lambda_i = \mathrm{diag}\{\lambda_1^i, \lambda_2^i\}, \tag{9}$$

where $\lambda_{1,2}^i$ denote the eigenvalues of A_i, namely,

$$\lambda_{1,2}^i = \begin{cases} \frac{1}{2}(\mathrm{tr}(A_i) \pm \sqrt{\Delta_{A_i}}), & \Delta_{A_i} > 0 \\ \frac{1}{2}(\mathrm{tr}(A_i) \pm \mathrm{j}\sqrt{-\Delta_{A_i}}), & \Delta_{A_i} < 0. \end{cases}$$

The change of coordinates that renders A_i diagonal is represented by V_i, i.e., $\Lambda_i = V_i^{-1}A_iV_i$.

Lemma 4. *There exist* V_1 *and* V_2 *constructed such that*

$$\|V_1^{-1}V_2\|\,\|V_2^{-1}V_1\| = \begin{cases} 1, & \det([A_1, A_2]) > 0 \\ |F|, & \det([A_1, A_2]) < 0. \end{cases} \tag{10}$$

Its proof is postponed to Appendix.

Remark 1. As will be seen in proving Proposition 1, $\|V_1^{-1}V_2\|$ $\|V_2^{-1}V_1\|$ actually accounts for the accumulation of the overshoots yielded by changing subsystems successively. In this sense, when $\det([A_1, A_2]) > 0$, there is no such overshoot at all.

Proposition 1. *Assume* $\det([A_1, A_2]) < 0$. *If* $|F| \leq \frac{\rho(A_1)}{\rho(A_2)}$, *then*

$$\hat{\rho}(A_1, A_2) = \max\{\rho(A_1), \rho(A_2)\}. \tag{11}$$

If $|F| > \frac{\rho(A_1)}{\rho(A_2)}$, *we have*

$$\hat{\rho}(A_1, A_2) \leq \sqrt{\rho(A_1)\rho(A_2)|F|}. \tag{12}$$

Proof. Given a switching law σ, let $N_\sigma(m-1)$ denote the number of changing subsystems up to an instant $m-1$, and, moreover, associate with σ two subsets of $\{1, 2, \ldots, N_\sigma(m-1)\}$, $\chi_1^\sigma(m-1)$ and $\chi_2^\sigma(m-1)$, which are composed of the sequence of changing to the first subsystem and the second one, respectively. Clearly, they are complementary to each other. To each $d \in \chi_p^\sigma(m-1)$, a positive integer l_d is assigned to indicate the length of the product of A_p with itself, namely, A_p is switched into activity at the instant $\sum_{i=1}^{d} l_i + 1$ till the instant $\sum_{i=1}^{d+1} l_i$. Given a sufficiently large integer m, let $s = N_\sigma(m-1)$, we have

$$\|A_{\sigma(m-1)}\cdots A_{\sigma(1)}A_{\sigma(0)}\|^{\frac{1}{m}} = \|A_p^{l_s}A_{3-p}^{l_{s-1}}\cdots A_{3-p}^{l_2}A_p^{l_1}\|^{\frac{1}{m}}, \tag{13}$$

where p equals either 1 or 2 and the integers $l_i \geq 1$ $(1 \leq i \leq s)$ satisfy $\sum_{i=1}^{s} l_i = m$. Obviously, $s \leq m$ and the distributions of $l_1, l_2, \ldots, l_s$ are uniquely determined by the specified σ.

For the sake of statement convenience, with a slight abuse of notation, we temporally redefine $A_1 = A_1/\rho(A_1)$ and $A_2 = A_2/\rho(A_1)$. Since $\mathcal{K}_{(\alpha_1A_1)(\alpha_2A_2)} = \mathrm{sgn}(\alpha_1\alpha_2)\mathcal{K}_{A_1A_2}$ for some scalars α_i, this rescaling does not influence $\mathcal{K}_{A_1A_2}$ but merely makes the spectral radii of A_1 and A_2 equal 1 and $\rho(A_2)/\rho(A_1)$, respectively. As a consequence, for the diagonal forms Λ_1 and Λ_2, which are described as in (9), one has that

$$\|\Lambda_1\| = 1 \quad \text{and} \quad \|\Lambda_2\| = \rho(A_2)/\rho(A_1). \tag{14}$$

Combining (10) and (14), from (13) we further deduce that

$$\begin{aligned} &\|A_{\sigma(m-1)}\cdots A_{\sigma(1)}A_{\sigma(0)}\|^{\frac{1}{m}} \\ &= \|(V_p\Lambda_p^{l_s}V_p^{-1})(V_{3-p}\Lambda_{3-p}^{l_{s-1}}V_{3-p}^{-1})\cdots(V_p\Lambda_p^{l_1}V_p^{-1})\|^{\frac{1}{m}} \\ &= \|V_p\Lambda_p^{l_s}(V_p^{-1}V_{3-p})\Lambda_{3-p}^{l_{s-1}}(V_{3-p}^{-1}V_p)\cdots(V_{3-p}^{-1}V_p)\Lambda_p^{l_1}V_p^{-1}\|^{\frac{1}{m}} \\ &\leq c^{\frac{1}{m}}\left[\prod_{i\in\chi_1^\sigma(m)}\|\Lambda_1^{l_i}\|\prod_{i\in\chi_2^\sigma(m)}\|\Lambda_2^{l_i}\|\right]^{\frac{1}{m}} \end{aligned}$$