Systems & Control Letters 96 (2016) 11–14

Contents lists available at ScienceDirect

Systems & Control Letters

journal homepage: www.elsevier.com/locate/sysconle

Stabilization of collocated systems by nonlinear boundary control

Ruth Curtain *, Hans Zwart

Bernoulli Institute for Mathematics and Computer Science, University of Groningen, P.O. Box 407, 9700AK Groningen, The Netherlands
Department of Applied Mathematics, University of Twente, P.O. Box 217, 7500AE Enschede, The Netherlands
Dynamics and Control group, Eindhoven University of Technology, P.O. Box 513, 5600MB Eindhoven, The Netherlands

ARTICLE INFO

Article history:
Received 3 October 2015
Received in revised form
23 June 2016
Accepted 27 June 2016

Keywords:
Semilinear differential equations
Partial differential equations
Dissipative systems
Boundary control

ABSTRACT

We design control laws for a class of dissipative infinite-dimensional systems using nonlinear boundary control action. The applications are to saturated control of SCOLE-type models.

1. Introduction and motivation

A popular way of stabilizing beam-like models is to apply collocated control and measurement action. In many cases, by a suitable choice of an extended state space, even collocated systems with boundary control action can be formulated as systems with bounded control. For examples of this formulation see Slemrod [1] and the examples in Oostveen [2, Chapter 9]. This results in a linear system

$$\dot{z}(t) = Az(t) + Bu(t), \qquad y(t) = B^*z(t), \qquad z(0) = z_0, \quad t \geq 0$$

on the state space Z, where A the infinitesimal generator of a contraction C_0-semigroup, $B \in \mathcal{L}(U, Z)$, and U is another Hilbert space. To illustrate the formulation of boundary control as bounded control in an appropriate space we recall the example used in [1]. It was motivated by the SCOLE-related model[1] from Bailey and Hubbard [3] of one of the arms of a satellite, consisting of a central hub with four flexible beams attached to it.

Example 1.1. This model describes the transverse vibrations of a beam of length L, which is clamped at one end and to which a point mass m is attached at the tip. Here $w(x, t)$ denotes the displacement of the beam and $u(t)$ is a scalar control. A piezoelectric film is bonded to the beam of length L, which applies a bending moment to the beam when a voltage is applied to it. This voltage is the control input of the system and the angular velocity at the tip is the measurement. The mathematical formulation is given by

$$\frac{\partial^2 w}{\partial t^2}(x,t) + \frac{\partial^4 w}{\partial x^4}(x,t) = 0 \quad \text{for } 0 < x < L, \tag{1}$$

with boundary conditions

$$w(0,t) = \frac{\partial w}{\partial x}(0,t) = 0, \tag{2}$$

$$m\frac{\partial^2 w}{\partial t^2}(L,t) = \frac{\partial^3 w}{\partial x^3}(L,t), \tag{3}$$

$$J\frac{\partial^3 w}{\partial t^2 \partial x}(L,t) = -\frac{\partial^2 w}{\partial x^2}(L,t) + u(t), \tag{4}$$

and measurement

$$y(t) = \frac{\partial^2 w}{\partial t \partial x}(L,t), \tag{5}$$

where m is the mass and J is its moment of inertia. In the rest of the paper we assume that both are equal to one. Let $\boldsymbol{L}_2(0, L)$ denote the linear space of square integrable functions on the interval $[0, L]$ and let $H^2(0, L)$ and $H^4(0, L)$ denote the standard Sobolev spaces,

* Corresponding author at: Bernoulli Institute for Mathematics and Computer Science, University of Groningen, P.O. Box 407, 9700AK Groningen, The Netherlands.
E-mail address: R.F.Curtain@rug.nl (R. Curtain).

[1] NASA Spacecraft Laboratory Control Experiment.

http://dx.doi.org/10.1016/j.sysconle.2016.06.014
0167-6911/

i.e.,

$$H^2(0,L) := \left\{ h \in \boldsymbol{L}_2(0,L) \mid h, \frac{dh}{dx} \text{ are abs. continuous}, \frac{d^2h}{dx^2} \in \boldsymbol{L}_2(0,L) \right\}$$

$$H^4(0,L) := \left\{ h \in \boldsymbol{L}_2(0,L) \mid h, \frac{dh}{dx}, \frac{d^2h}{dx^2}, \frac{d^3h}{dx^3} \text{ are abs. continuous}, \frac{d^4h}{dx^4} \in \boldsymbol{L}_2(0,L) \right\}.$$

Define the state space by

$$Z = \left\{ h = \begin{pmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \end{pmatrix} \in H^2(0,L) \times \boldsymbol{L}_2(0,L) \times \mathbb{R} \times \mathbb{R} \mid h_1(0) = \frac{dh_1}{dx}(0) = 0 \right\}$$

with the inner product

$$\langle h, \tilde{h} \rangle_Z = \int_0^L \frac{d^2h_1}{dx^2}(x)\frac{d^2\tilde{h}_1}{dx^2}(x)dx + \int_0^L h_2\tilde{h}_2 dx + h_3\tilde{h}_3 + h_4\tilde{h}_4.$$

The input space and output space are $U = \mathbb{R}$. We define the operators $A : \boldsymbol{D}(A) \subset Z \to Z$ and $B \in \mathcal{L}(U,Z)$ by

$$\boldsymbol{D}(A) = \left\{ h \in H^4(0,L) \times H^2(0,L) \times \mathbb{R} \times \mathbb{R} \mid h_1(0) = \frac{dh_1}{dx}(0) = 0, h_2(0) = \frac{dh_2}{dx}(0) = 0,\ h_2(L) = h_3,\ \frac{dh_2}{dx}(L) = h_4 \right\},$$

$$A = \begin{pmatrix} 0 & I & 0 & 0 \\ -\frac{d^4}{dx^4} & 0 & 0 & 0 \\ \frac{d^3}{dx^3}|_{x=L} & 0 & 0 & 0 \\ -\frac{d^2}{dx^2}|_{x=L} & 0 & 0 & 0 \end{pmatrix}, \quad Bu = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} u.$$

When

$$z_1(t) = w(t), \quad z_2(t) = \dot{w}(t), \quad z_3(t) = \frac{\partial w}{\partial t}(L,t), \quad z_4(t) = \frac{\partial^2 w}{\partial t \partial x}(L,t).$$

Eqs. (1)–(5) are equivalent to

$$\dot{z}(t) = Az(t) + Bu(t), \quad z_0(0) = z_0,$$
$$y(t) = B^* z(t).$$

In Slemrod [1] it is shown that B is bounded, A is skew-adjoint with compact resolvent, and $\Sigma(A, B, B^*, 0)$ is approximately observable. In particular, A generates a unitary group on Z. ■

It is well-known that under mild conditions the output feedback action $u(t) = -y(t)$ will produce an asymptotically stable closed-loop system. We shall use the following formulation from Oostveen [2, Chapter 2].

Theorem 1.2. *Let Z, U be Hilbert spaces, $B \in \mathcal{L}(U,Z)$ and A the infinitesimal generator of a contraction C_0-semigroup. Assume that A has compact resolvent, and the state linear system $\Sigma(A, B, B^*, 0)$ is approximately controllable. Then*

a. *for all $\kappa > 0$, the operator $A - \kappa BB^*$ generates a strongly stable semigroup, $T_{-\kappa BB^*}(t)$;*
b. *the closed-loop system $\Sigma(A - \kappa BB^*, B, B^*, 0)$ is input stable, i.e.,*
$$\int_0^\infty \|T_{-\kappa BB^*}(s)Bu(s)\|^2 ds \le \frac{1}{2}\|u\|^2_{\boldsymbol{L}_2(0,\infty)}, \quad u \in \boldsymbol{L}_2(0,\infty);$$
c. *for all $u \in \boldsymbol{L}_2(0,\infty)$ we have*
$$\int_0^t T_{-\kappa BB^*}(t-s)u(s)ds \to 0 \quad \text{as } t \to \infty.$$

So the feedback $u(t) = -y(t)$ in the above example would produce a strongly stable system. However, in practice, the control action that one can apply is often limited by physical constraints. An example is when the magnitude of the control is constrained to be bounded. Slemrod [1] considered the case of saturated boundary control: $u(t) = -y(t)\chi(y(t))$, where

$$\chi(y) = \begin{cases} 1, & \|y\| < 1 \\ \frac{1}{\|y\|}, & \|y\| \ge 1. \end{cases} \tag{6}$$

This results in the following closed-loop semilinear system with a uniform Lipschitz nonlinearity:

$$\dot{z}(t) = Az(t) - BB^*z(t)\chi(B^*z(t)), \quad z(0) = z_0. \tag{7}$$

His main theoretical result [1, Theorem 7.1] was the following.

Theorem 1.3. *Let Z be a Hilbert space, $B \in \mathcal{L}(\mathbb{R}, Z)$ and A the infinitesimal generator of a contraction semigroup. Assume that A has compact resolvent. Then the semilinear differential equation (7) has a global mild solution $z(t; z_0)$ and if $\Sigma(A, B, B^*, 0)$ is approximately observable, i.e.,*

$$B^*T(t)z_0 = 0 \quad \text{for all } t \ge 0 \Longrightarrow z = \{0\},$$

then $z(t; z_0) \to 0$ as $t \to \infty$.

Slemrod used a Lyapunov approach and several abstract results from the theory of nonlinear contraction semigroups. Using a system theoretic approach, we obtain a more general result with much shorter proofs. In our main result, Theorem 2.2, we allow for a significantly larger class of nonlinearities and a general Hilbert space for the input space.

2. Stabilization of collocated systems

Throughout the paper we consider the collocated state linear system $\Sigma(A, B, B^*, 0)$ on a real Hilbert space. In the formulation of Theorems 1.2 and 1.3 we see that the first assumes approximately controllability whereas the second assumes approximately observability.

We show that under certain assumptions on A either condition will ensure that $A - \kappa BB^*$ generates a strongly stable semigroup.

Theorem 2.1. *Let Z, U be Hilbert spaces, $B \in \mathcal{L}(U,Z)$ and A the infinitesimal generator of a contraction C_0-semigroup. Assume that A has compact resolvent, and that the state linear system $\Sigma(A, B, B^*, 0)$ is approximately controllable or approximately observable. Then $A - \kappa BB^*$ and $A^* - \kappa BB^*$ generate strongly stable semigroups for all $\kappa > 0$.*

Proof. If the system is approximately controllable, then the assertion follows from Theorem 1.2. So it remains to show that the assertion also holds when the system $\Sigma(A, B, B^*, -)$ is approximately observable. By [4, Section 4.1] we have that the dual system $\Sigma(A^*, B, B^*, -)$ is approximately controllable, and since A^*