Systems & Control Letters 96 (2016) 30–36

Contents lists available at ScienceDirect

Systems & Control Letters

journal homepage: www.elsevier.com/locate/sysconle

Gradient and passive circuit structure in a class of non-linear dynamics on a graph

Herbert Mangesius [a,*], Jean-Charles Delvenne [b], Sanjoy K. Mitter [c]

[a] Department of Electrical and Computer Engineering, Technische Universität München, Arcisstrasse 21, D-80209 Munich, Germany
[b] ICTEAM and CORE, Université catholique de Louvain, 4 Avenue Lemaître, B-1348 Louvain-la-Neuve, Belgium
[c] Laboratory for Information and Decision Systems, Department of Electrical Engineering and Computer Science, MIT, Cambridge, MA 02139, USA

ARTICLE INFO

Article history:
Received 30 June 2015
Received in revised form
16 May 2016
Accepted 24 June 2016

Keywords:
Dynamics on a graph
Gradient flows
Passive circuits
Information inequalities
Lyapunov functions

ABSTRACT

We consider a class of non-linear dynamics on a graph that contains and generalizes various models from network systems and control and study convergence to uniform agreement states using gradient methods. In particular, under the assumption of detailed balance, we provide a method to formulate the governing ODE system in gradient descent form of sum-separable energy functions, which thus represent a class of Lyapunov functions; this class coincides with Csiszár's information divergences. Our approach bases on a transformation of the original problem to a mass-preserving transport problem and it reflects a little-noticed general structure result for passive network synthesis obtained by B.D.O. Anderson and P.J. Moylan in 1975. The proposed gradient formulation extends known gradient results in dynamical systems obtained recently by M. Erbar and J. Maas in the context of porous medium equations. Furthermore, we exhibit a novel relationship between inhomogeneous Markov chains and passive non-linear circuits through gradient systems, and show that passivity of resistor elements is equivalent to strict convexity of sum-separable stored energy. Eventually, we discuss our results at the intersection of Markov chains and network systems under sinusoidal coupling.

1. Motivation

Gradient methods provide an elegant way to physics motivated modeling [1,2] and are closely linked to passivity theory and the circuit concept [3,4]. They are a basic tool in studying and designing non-linear systems on a graph, e.g., in distributed optimization [5] or in multi-robot problems such as coverage or formation control, cf., e.g., [6,7], and references therein.

Another pillar in network system studies is the classical consensus problem [8]. An equivalence between the dynamics (trajectories) of Markov chains and consensus networks has been source of recent advances in consensus theory [9]. For LTI symmetric consensus networks such an equivalence has been linked to the averaging dynamics of unit capacitor RC circuits in [6, chap. 3]. Within the mathematics community, a static relationship is usually considered between Markov chains and electric circuits (resistor networks) [10]. The static (algebraic) circuit equations due to Kirchhoff and Ohm in fact are known to serve as generic structure underlying various scientific and computational problems, see, e.g., [11, chap. 2].

Gradient formulations of Markov chains using sum-separable energy functions have been of recent interest in dynamical and non-linear systems [12–15]. Interestingly, sum-separability of energy is an axiom in interconnected dissipative systems [4] and has origins in the circuit concept.

In this paper we bring these various concepts together in novel ways, based on a gradient structure for a class of non-linear dynamics on a graph that covers a wide range of prominent network system problems.

2. Problem description and related literature

Let $\mathsf{G} = (N, B, w)$ be a weighted directed graph, where $N = \{1, 2, \ldots, n\}$ is the set of nodes, $B = \{1, 2, \ldots, b\} \subseteq N \times N$ denotes the set of branches whose elements are ordered pairs (j, i) denoting an edge from node j to i, and $w : B \to \mathbb{R}_{>0}$ is a weighting function, such that $w((j, i)) =: w_{ij}$, if $(j, i) \in B$, else $w_{ij} = 0$. Associated to a graph is the Laplace matrix $\mathbf{L}$, defined component-wise as $[\mathbf{L}]_{ij} = -w_{ij}$, $[\mathbf{L}]_{ii} = \sum_j w_{ij}$. For strongly connected graphs, denote the positive left-eigenvector associated

* Corresponding author.
E-mail address: mangesius@tum.de (H. Mangesius).

http://dx.doi.org/10.1016/j.sysconle.2016.06.019
0167-6911/

to the unique zero eigenvalue of the Laplacian by $\boldsymbol{c}$, and define $\mathbf{C} := \mathrm{diag}\{c_1, c_2, \ldots, c_n\}$.

An important generalization of the symmetry condition on Laplacians that $\mathbf{L} = \mathbf{L}^\top$ is the particular type-symmetry that for some $\mathbf{C}$, and $i, j \in N$,

$$c_i w_{ij} = c_j w_{ji} \Leftrightarrow \mathbf{CL} = \mathbf{L}^\top \mathbf{C}. \tag{1}$$

Eq. (1) is known in the literature on Markov chains as detailed balance, or as reversibility w.r.t. $\boldsymbol{c}$, cf., [16, chap. 2].

We consider the general class of dynamics on a graph G described component-wise by an ODE of the type

$$\dot{x}_i = \sum_{j:(j,i)\in B} w_{ij}\, \phi(x_j, x_i), \quad i \in N, \tag{2}$$

where $\phi(\cdot,\cdot)$ is Lipschitz continuous, $\phi(a, b)$ negative if $a < b$, zero iff $a = b$, positive if $a > b$, and $|\phi(a, b)|$ is increasing if $|a - b|$ is increasing.

The class (2) includes many known network models: The usual linear consensus system [8] is obtained from setting $\phi(x_j, x_i) = x_j - x_i$. If $\phi(x_j, x_i) = f(x_j - x_i)(x_j - x_i)$, with $f(z) = f(-z) > 0$, then, the ODE (2) describes a continuous-time opinion dynamics [17]. For instance, one may choose $f(z) = |\tanh(p\cdot z)|, p > 0$, which is a good choice for modeling saturation phenomena in the interaction. If $\phi(x_j, x_i) = \psi(x_j - x_i)$, $\psi(z) = -\psi(z)$, then we recover the non-linear consensus class introduced by Olfati-Saber and Murray in [18], with $\psi = \sin$ a prominent instance. Beyond the presented known interaction types, our model also includes couplings of the form $\phi(x_j, x_i) = g(x_j) - g(x_i)$, where g is an increasing function,[1] e.g., $\ln(x)$, e^x, x^p, $p > 0$, on the respective domain of definition. The latter interaction type covers a discrete version of an equation system that models the non-linear diffusion of a gas in porous media, see [12] (and [19] for the continuous context).

From an operational point of view, we are interested in bringing the ODE system (2), under the assumption of detailed balance, into the gradient form

$$\dot{\boldsymbol{q}} = -\mathbf{K}(\boldsymbol{q})\nabla E(\boldsymbol{q}), \tag{3}$$

where $\boldsymbol{q}$ is a suitable transform of the original state $\boldsymbol{x}$, $\mathbf{K}(\cdot)$ is a symmetric, positive semi-definite matrix function that inherits the sparsity structure of the graph G, and $E(\boldsymbol{q})$ is a sum-separable Lyapunov function.

This structure defines gradient descent systems living on subspaces of $\mathbb{R}^n$, where $-\nabla E(\cdot)\cdot \mathbf{K}(\cdot)\nabla E(\cdot)$, describing locally the dissipation rate of E, is negative definite. In the context of gradient systems this structure is quite particular, as we impose the sparsity constraint given by G, require sum-separability of the potential E, and do not require positive definiteness of the inverse metric $\mathbf{K}$; these constraints are not usual from a classical gradient system point of view, cf., e.g., [20], but turn out to be elementary in a passive circuits context.

For particular cases of graph weightings and functions ϕ, gradient structures for the class (2) have been established: For symmetric consensus systems, (i.e. $\mathbf{L} = \mathbf{L}^\top$, $\phi(x_j, x_i) = x_j - x_i$), it is well known that the network dynamics are a gradient descent of the (non-sum-separable) interaction potential $\frac{1}{2}\boldsymbol{x}^\top \mathbf{L}\boldsymbol{x}$, see, e.g., [8]. In [21] a port-Hamiltonian view as gradient descent of the sum-of-squares energy $\frac{1}{2}\sum_{i\in N} x_i^2$ is presented. Under the less restrictive assumption of detailed balance weightings, the linear system dynamics (understood as Markov chain) has been formulated as gradient descent of free energy, resp. of relative entropy, in the works [15,14]. In [12], for systems with detailed balance weighting, and $\phi(x_j, x_i) = g(x_j) - g(x_i)$, g increasing, a smooth gradient descent structure is presented for sum-separable energies $\sum_{i\in N} c_i H(x_i)$, H being strictly convex and smooth on $\mathbb{R}_{>0}$. For the particular non-separable interaction case of having sinusoidal coupling, but symmetric weighting, gradient flow structures are represented, e.g., in [22] or [23], where energy functions however are non-separable.

In the following we solve the general gradient representation problem and motivate the proposed structure requiring sum-separable energy functions from a passivity and circuit systems viewpoint.

3. Gradient representation

With the following result we provide a procedure to bring a dynamics (2) into the form (3). By that we characterize a family of sum-separable Lyapunov functions characterizing asymptotic stability of agreement states, i.e., states where all components are equal.

Theorem 1. *Consider a network system dynamics governed by the protocol (2) on a strongly connected graph G such that detailed balance (1) holds for some $\mathbf{C}$. Define the new state $\boldsymbol{q} := \mathbf{C}\boldsymbol{x}$, and consider the sum-separable function*

$$E(\boldsymbol{q}) := \sum_{i\in N} c_i H(c_i^{-1} q_i), \tag{4}$$

where $H : \mathbb{R} \to \mathbb{R}$ is any $\mathscr{C}^2$-function, and set $h(z) := \frac{\mathrm{d}H(z)}{\mathrm{d}z}$. If H is strictly convex, then the system can be represented as

$$\dot{\boldsymbol{q}} = -\mathbf{K}(\boldsymbol{q})\nabla E(\boldsymbol{q}),$$

where $\mathbf{K}(\cdot)$ is defined as the irreducible and symmetric Laplace matrix having components

$$[\mathbf{K}]_{ij} := \begin{cases} -c_i w_{ij} \dfrac{\phi(x_j, x_i)}{h(x_j) - h(x_i)} & \text{if } j \neq i, \\ -\displaystyle\sum_{k=1,k\neq i}^{n} [\mathbf{K}]_{ik} & \text{if } i = j. \end{cases}$$

The function (4) is a Lyapunov function establishing asymptotic stability of the equilibrium point $x_\infty \mathbf{1}$, with equilibrium value the weighted arithmetic mean $x_\infty = \frac{\sum_{i\in N} c_i x_i(0)}{\sum_{i\in N} c_i}$.

Proof. First, we observe that by the chain rule with $c_i^{-1} q_i = x_i$,

$$\frac{\partial}{\partial q_i} E(\boldsymbol{q}) = c_i \frac{\partial H(x_i)}{\partial x_i}\frac{\partial x_i}{\partial q_i} = c_i h(x_i) c_i^{-1} = h(x_i).$$

The network dynamics (2) can be written equivalently as

$$\frac{1}{c_i} c_i \dot{x}_i = \sum_{j:(j,i)\in B} w_{ij}\phi(x_j, x_i) \Leftrightarrow \dot{q}_i = \sum_{j:(j,i)\in B} c_i w_{ij}\phi(x_j, x_i).$$

Expanding by $h(x_j) - h(x_i)$ yields

$$\begin{aligned} \dot{q}_i &= \sum_{j:(j,i)\in B} c_i w_{ij} \frac{\phi(x_j, x_i)}{h(x_j) - h(x_i)} \big(h(x_j) - h(x_i)\big) \\ &= \sum_{j:(j,i)\in B} [\mathbf{K}]_{ij} \left(\frac{\partial}{\partial q_j} E(\boldsymbol{q}) - \frac{\partial}{\partial q_i} E(\boldsymbol{q})\right) \end{aligned}$$

$$\Leftrightarrow \dot{\boldsymbol{q}} = -\mathbf{K}(\boldsymbol{q})\nabla E(\boldsymbol{q}),$$

where we use the identity $\sum_{j:(j,i)\in B} [\mathbf{K}]_{ij} = -[\mathbf{K}]_{ii}$.

Next, we show that the matrix $\mathbf{K}$ is a symmetric, irreducible Laplace matrix. As H is strictly convex and of type $\mathscr{C}^2$, h, as derivative of H, is an increasing function, i.e., for any two real numbers a, b, $h(a) < h(b)$, whenever $a < b$. We observe that

[1] Or $\phi(x_j, x_i) = l(x_i) - l(x_j)$, where l is a decreasing function.