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Eigenvalue clustering, control energy, and logarithmic capacity

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ABSTRACT

We prove two bounds showing that if the eigenvalues of a matrix are clustered in a region of the complex plane then the corresponding discrete-time linear system requires significant energy to control. A curious feature of one of our bounds is that the dependence on the region is via its logarithmic capacity, which is a measure of how well a unit of mass may be spread out over the region to minimize a logarithmic potential.

1. Introduction

We will consider discrete-time linear systems

$$x(t+1) = Ax(t) + Bu(t), \tag{1}$$

where $A \in \mathbb{C}^{n\times n}$ and $B \in \mathbb{C}^{n\times k}$. Our goal is to understand the relation between the locations of the eigenvalues of A within the complex plane and the energy needed to steer Eq. (1) by choosing the input $u(t)$. We will prove two bounds to the effect that if the eigenvalues of A are clustered, then Eq. (1) is "difficult to control" in the sense of requiring large inputs to steer between states.

Our work is related to a growing body of literature investigating the control properties of large-scale systems. A strand of this literature, to which this paper belongs, is to identify the fundamental limitations for controlling such networks [1–8]. A common concern is control difficulty when n (the number of states) is large; it has been experimentally observed that in some scenarios the minimum control energy grows exponentially as a function of n [8,9]. In this note, we will study how eigenvalue locations of A in the complex plane can sometimes be a de-facto obstacle to efficient control for systems with many states.

It is textbook material that the energy needed to steer a linear system is related to the smallest eigenvalue of the controllability Gramian, and we spell this out before describing the problem and our results. Given initial state x_0 and final state x_f, we let $\mathcal{E}(A, B, x_0 \to x_f, t)$ be the minimal energy $\sum_{i=0}^{t-1} \|u(i)\|_2^2$ among all inputs which result in $x(t) = x_f$ starting from $x(0) = x_0$. We then use this notion to define the "difficulty of controllability" of a linear system by considering the worst-case energy needed to move from the origin to a point on the unit sphere, i.e.,

$$\mathcal{E}(A, B, t) := \sup_{\|y\|_2=1} \mathcal{E}(A, B, 0 \to y, t).$$

We will allow both sides to be infinite if there is a vector y on the unit sphere which cannot be reached with any choice of $u(0), \ldots, u(t-1)$. This is not the only way to formalize the difficulty of controllability of a linear system (for example, one might also consider the expected energy to move to a random point on the unit sphere) but it is among the most natural. Defining the t-step controllability Gramian as

$$W(t) := \sum_{i=0}^{t} A^i BB^* (A^*)^i, \tag{2}$$

basic linear algebra then gives that

$$\mathcal{E}(A, B, t) = \frac{1}{\lambda_{\min}(W(t-1))},$$

where $\lambda_{\min}(W(t-1))$ is the smallest eigenvalue of the nonnegative definite matrix $W(t-1)$.

Thus the question of how difficult a system is to control (in a certain worst-case sense) reduces to the analysis of the smallest eigenvalue of the controllability Gramian. The study of that eigenvalue is the subject of this note.

We are motivated by a recent result of [5] which showed that if A is a diagonalizable matrix with m eigenvalues within the circle

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$\mathcal{X} = \{z \mid |z| \le \mu < 1\}$, then the smallest eigenvalue of $W(t)$ for any t is upper bounded by a product of two terms one of which is $\mu^{2(\lceil m/k\rceil -1)}/(1-\mu^2)$ (where recall k is the number of columns of B). In other words, if m is large enough compared to k and μ is not close to 1, then the smallest eigenvalue of $W(t)$ is exponentially close to zero.

Our goal here is to produce similar results for other sets $\mathcal{X}$, especially those which are not contained within the interior of the unit circle. In this case we will not be able to obtain bounds on $\lambda_{\min}(W(t))$ for all t, but we will be able to upper bound this eigenvalue for some concrete choices of time t.

1.1. Related work

As already mentioned, the main motivating work for the present paper is [5], which was the first (to our knowledge) to obtain results connecting eigenvalue clustering to lower bounds on control energy. Follow-up work included [7], which studied the relation between the difficulty of controllability and the propagation of inputs by the system in various directions, and [6] which studied connections to measures of centrality such as PageRank.

The existence of small eigenvalues of the discrete-time controllability Gramian appears to have not attracted significant attention in the existing control literature beyond the above papers. In continuous time, results on the condition number (which is the ratio of the largest and smallest eigenvalue of the Gramian) in the case when A is stable have been derived [10], as well as more general results on ratios of eigenvalues [11,12].

Our work is also related to a series of recent preprints analyzing properties of eigenvalues of the Gramian of linear [2–4] and bilinear [1] systems. These papers studied the efficiency of algorithms for placement of sensors and actuators, as well as the underlying properties of the Gramian that allow for efficient approximation algorithms.

1.2. Our results

This note has two main results. The first concerns control energy at the first time the system can become controllable, namely at time

$$t_{\min} := \lceil n/k \rceil - 1,$$

where $\lceil x \rceil$ is the smallest integer which is at least x and k, recall, is the number of columns of B. It is immediate that if $t < t_{\min}$, then $W(t)$ is singular because there are not enough columns for the controllability matrix to be full-rank. Our first result shows that if the eigenvalues of A lie in a set with logarithmic capacity (to be formally defined later) smaller than one, then $W(t_{\min})$ has an eigenvalue upper bounded by something that decays to zero exponentially fast in $t_{\min}$.

Our second result considers A which are Hermitian with m eigenvalues which are stable (i.e., which lie in $[-1, 1]$). Roughly speaking, we show that if $t = O\left(\left(\frac{m}{k}\right)^{2-\epsilon}\right)$ for[1] some $\epsilon > 0$, the controllability Gramian $W(t)$ has an eigenvalue which is upper bounded by a quantity that goes to zero as $m/k \to +\infty$. For example, if $t = O\left((m/k)^{3/2}\right)$, our result gives the bound $\lambda_{\min}(W(t)) = O\left((m/k)^{3/2}\, e^{-\sqrt{m/k}}\right)$ in this case.

The formal statements of these results are a little involved and will be given within the body of this paper. We conclude our summary by illustrating their use on some simple examples. Suppose $x(t+1) = Ax(t) + bu(t)$ where $A \in \mathbb{C}^{n\times n}$ is diagonalizable as $VAV^{-1} = D$ and $b \in \mathbb{R}^{n\times 1}$ is a vector of unit norm. Then:

- If the eigenvalues of A are contained within any equilateral triangle of side length 2 in the complex plane, then there is some n_0 such that for all $n \ge n_0$, we have

$$\lambda_{\min}(W(n-1)) \le \|V\|_2^2 \|V^{-1}\|_2^2 \cdot 0.133^n. \quad (3)$$

  By contrast, if all the eigenvalues are contained within a circle in the complex plane of the very same area as this equilateral triangle, our methods give the bound

$$\lambda_{\min}(W(n-1)) \le \|V\|_2^2 \|V^{-1}\|_2^2 \cdot 0.552^n, \quad (4)$$

  once again for n large enough.
- Suppose $n = 10{,}000$ and A is a Hermitian matrix at least half of whose eigenvalues are stable. It turns out that this is enough information to conclude that

$$\lambda_{\min}(W(250{,}000)) \le 1.03 \times 10^{-37} \quad (5)$$

$$\lambda_{\min}(W(1{,}000{,}000)) \le 1.58 \times 10^{-4}. \quad (6)$$

  In other words, the presence of many stable modes appears to be a significant obstacle to the efficient control of Hermitian systems.

1.3. Organization of this paper

We conclude the introduction with Section 1.4 which introduces some notation as well as some background facts which we will draw on throughout this paper. Section 2 is dedicated to proving our first main result, namely the bound on $W(t_{\min})$ in terms of logarithmic capacity. Section 3 proves our second main result, which bounds the eigenvalues of $W(t)$ for a range of times t in the special case when the matrix A is Hermitian with many stable eigenvalues. We end with some concluding remarks in Section 4.

1.4. Notation and background

We first describe some notation that we will use for the remainder of the paper. We use the standard notation $o_l(1)$ to denote any function of l that goes to zero as $l \to +\infty$. Given a matrix V, its condition number is defined as $\mathrm{cond}(V) := \|V\|_2 \|V^{-1}\|_2$. The Frobenius norm of V is denoted as $\|V\|_F$. As is standard, V^* will denote the conjugate transpose of V and $\overline{V}$ will denote its (elementwise) complex conjugate. We will use $\mathcal{P}_j$ to denote the set of univariate polynomials with complex coefficients of degree at most j, and $\mathcal{P}_j'$ to denote the set of *monic*[2] univariate polynomials with complex coefficients of degree j.

Given a compact set K in the complex plane, let μ^K be the set of probability measures on K, i.e., the set of Borel measures μ supported on K which satisfy $\mu(K) = 1$. We then define $I(K)$, called the logarithmic energy of the set K, as

$$I(K) := \sup_{\mu \in \mu^K} \int_{K\times K} \log|z-w|\, \mathrm{d}\mu(z)\, \mathrm{d}\mu(w).$$

The logarithmic capacity is then defined as

$$\mathrm{cap}(K) := e^{I(K)}.$$

Logarithmic capacity comes up in our results due to its connection with polynomial approximation, which we now describe. Given a set $\mathcal{X} \subset \mathbb{C}$, we define

$$\mathrm{Err}(l, \mathcal{X}) := \min_{p\in\mathcal{P}_{l-1}} \max_{z\in X} \left|z^l - p(z)\right|. \quad (7)$$

[1] We use the standard O-notation, i.e., the statement $f = O(g)$ where f and g are positive quantities denotes the existence of a constant C such that $f \le Cg$.

[2] Meaning the coefficient in front of the highest power is one.