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# Fast diffeomorphic matching to learn globally asymptotically stable nonlinear dynamical systems

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ABSTRACT

We propose a new diffeomorphic matching algorithm and use it to learn nonlinear dynamical systems with the guarantee that the learned systems have global asymptotic stability. For a given set of demonstration trajectories, and a reference globally asymptotically stable time-invariant system, we compute a diffeomorphism that maps forward orbits of the reference system onto the demonstrations. The same diffeomorphism deforms the whole reference system into one that reproduces the demonstrations, and is still globally asymptotically stable.



## 1. Introduction

We consider the problem of learning dynamical systems (DS) from demonstrations. More precisely, given a list of trajectories $(\mathbf{x}_i(t))$ observed as timed sequences of points in $\mathbb{R}^d$, the objective is to build a smooth autonomous system $\dot{\mathbf{x}} = f(\mathbf{x})$ (i.e. a vector field) that reproduces the demonstrations as closely as possible.

The ability to construct such DS is an important skill in imitation learning (see for example [1]). The learned systems can be used as dynamical movement primitives generating goal-directed behaviors [2].

Modeling movement primitives with DS is convenient for closed loop implementations, and their generalization to unseen parts of the state space provides robustness to spatial perturbations. Moreover, the choice of autonomous (i.e. time-invariant) systems, while not always suitable or preferable, is interesting in many situations as they are inherently robust to temporal perturbations.

The most common movement primitives consist of motions that converge towards a single targeted configuration. They correspond to globally asymptotically stable DS. But classical learning algorithms cannot provide the guarantee that their output is always globally asymptotically stable. They might produce DS with instabilities or spurious attractors. This issue has recently been studied by Khansari-Zadeh and Billard [3,4] who proposed several approaches to learn globally asymptotically stable nonlinear DS. One of the main ideas they investigated consists in learning a Lyapunov function candidate (or simply Lyapunov candidate[1]) $L$ that is highly compatible with the demonstrations in the following sense: at almost every point $\mathbf{x}_i(t_j)$, the estimated or measured velocity $\mathbf{v}_i(t_j)$ is such that its scalar product with the gradient of $L$ is negative: $\mathbf{v}_i(t_j) \cdot \nabla L(\mathbf{x}_i(t_j)) < 0$. Once $L$ is found, a learning algorithm optimizes a weighted sum of DS that admit $L$ as a common Lyapunov function, therefore ensuring the global asymptotic stability of the resulting DS. Alternatively, $L$ can be used to modify movement primitives by correcting trajectories whenever they would violate the compatibility condition.

The main limitation of this method comes from the difficulty to find good Lyapunov candidates. In SEDS (*Stable Estimator of Dynamical Systems*), one of the first algorithms proposed by Khansari-Zadeh and Billard, the Lyapunov function is set to be the $l^2$-norm squared ($\|\cdot\|^2$), which means that all trajectories produced by the learned DS are such that the distance to the target is monotonically decreasing. In their more recent algorithm CLF-DM (*Control Lyapunov Function-based Dynamic Movements*),

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[1] In this paper (cf. Definition 1), a Lyapunov function candidate is a $\mathcal{C}^1$ function from $\mathbb{R}^d$ to $\mathbb{R}_{\geq 0}$, radially unbounded, taking the value 0 at a target point $\mathbf{x}^*$ and with no other local extremum. We generally assume $\mathbf{x}^* = \mathbf{0}$.

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the search for a Lyapunov candidate is done among a set called Weighted Sums of Asymmetric Quadratic Functions (WSAQF). It highly increases the set of DS that can be learned, but the restrictions remain significant as the search is limited to a small convex subset of the set of Lyapunov candidates.

To go further, Neumann and Steil [5] suggested to initially compute a Lyapunov candidate with the above method, and then apply a simple diffeomorphism (of the form $\mathbf{x} \mapsto \eta(\mathbf{x})\mathbf{x}$, with $\eta(\mathbf{x}) \in \mathbb{R}_{\geq 0}$) that deforms the space and transforms the Lyapunov candidate into the function $\mathbf{x} \mapsto \|\mathbf{x}\|^2$, thus simplifying the trajectories of the demonstrations. In the deformed space, an algorithm like SEDS is then more likely to learn a globally asymptotically stable DS that reproduces faithfully the demonstrations.

In this paper, we propose a more direct diffeomorphism-based approach. Our contribution is twofold.

- First, we introduce a new algorithm for diffeomorphic matching (Sections 2 and 3) and show from experimental comparisons that it tends to be one or two orders of magnitude faster than a state-of-the-art algorithm.
- Then, we explain how it can be used to directly map simple trajectories of a DS like $\dot{\mathbf{x}} = -\mathbf{x}$ onto the trajectories of the training data (Section 4). This gives a new way to generate Lyapunov candidates as well as globally asymptotically stable smooth autonomous systems reproducing the demonstrations.

The most direct applications of this work are in motor control and robotics, but we believe that learning globally asymptotically stable nonlinear DS and computing Lyapunov candidates can be useful for various types of systems and control design problems.

## 2. Diffeomorphic locally weighted translations

Given a smooth (symmetric positive definite) kernel function $k_\rho(\mathbf{x}, \mathbf{y})$, depending on some parameter $\rho$, such that $\forall \mathbf{x}$, $k_\rho(\mathbf{x}, \mathbf{x}) = 1$ and $k_\rho(\mathbf{x}, \mathbf{y}) \to 0$ when $\|\mathbf{y}-\mathbf{x}\| \to \infty$, given a "direction" $\mathbf{v} \in \mathbb{R}^d$ and a "center" $\mathbf{c} \in \mathbb{R}^d$, we consider the following *locally weighted translation*:

$$\psi_{\rho,\mathbf{c},\mathbf{v}}(\mathbf{x}) = \mathbf{x} + k_\rho(\mathbf{x}, \mathbf{c})\mathbf{v}.$$

**Theorem 1.** *If* $\forall (\mathbf{x}, \mathbf{y}) \in \mathbb{R}^d \times \mathbb{R}^d$, $\frac{\partial k_\rho}{\partial \mathbf{x}}(\mathbf{x}, \mathbf{y}) \cdot \mathbf{v} > -1$, *then* $\psi_{\rho,\mathbf{c},\mathbf{v}}$ *is a smooth* ($\mathcal{C}^\infty$) *diffeomorphism.*

**Proof.** For a given $\mathbf{x} \in \mathbb{R}^d$, let us try to find $\mathbf{y} \in \mathbb{R}^d$ such that $\psi_{\rho,\mathbf{c},\mathbf{v}}(\mathbf{y}) = \mathbf{x}$. This can be rewritten $\mathbf{y} = \mathbf{x} - k_\rho(\mathbf{y}, \mathbf{c})\mathbf{v}$, so we know that $\mathbf{y}$ must be of the form $\mathbf{x} + r\mathbf{v}$. The equation becomes $\psi_{\rho,\mathbf{c},\mathbf{v}}(\mathbf{x} + r\mathbf{v}) = \mathbf{x}$, i.e.: $r\mathbf{v} + k_\rho(\mathbf{x} + r\mathbf{v}, \mathbf{c})\mathbf{v} = \mathbf{0}$. If $\mathbf{v} = \mathbf{0}$, $\psi_{\rho,\mathbf{c},\mathbf{v}}$ is the identity (and a smooth diffeomorphism), and $\mathbf{y} = \mathbf{x}$. Otherwise, solving $\psi_{\rho,\mathbf{c},\mathbf{v}}(\mathbf{y}) = \mathbf{x}$ amounts to solving $r + k_\rho(\mathbf{x} + r\mathbf{v}, \mathbf{c}) = 0$.

Let us define:

$$h_\mathbf{x} : r \in \mathbb{R} \mapsto r + k_\rho(\mathbf{x} + r\mathbf{v}, \mathbf{c}) \in \mathbb{R}.$$

If $\frac{\partial k_\rho}{\partial \mathbf{x}}(\mathbf{x}, \mathbf{c}) \cdot \mathbf{v} > -1$, we get: $\forall r \in \mathbb{R}$, $\frac{dh_\mathbf{x}}{dr}(r) > 0$. Because of the absolute monotonicity of $h_\mathbf{x}$, and since $h_\mathbf{x}(r)$ tends to $-\infty$ when $r$ tends to $-\infty$, and to $+\infty$ when $r$ tends to $+\infty$, we deduce that there exists a unique $s_{\rho,\mathbf{c},\mathbf{v}}(\mathbf{x}) \in \mathbb{R}$ such that $h_\mathbf{x}(s_{\rho,\mathbf{c},\mathbf{v}}(\mathbf{x})) = 0$. It follows that the equation $\psi_{\rho,\mathbf{c},\mathbf{v}}(\mathbf{y}) = \mathbf{x}$ has a unique solution: $\mathbf{y} = \mathbf{x} + s_{\rho,\mathbf{c},\mathbf{v}}(\mathbf{x})\mathbf{v}$. We conclude that $\psi_{\rho,\mathbf{c},\mathbf{v}}$ is invertible, and:

$$\psi^{-1}_{\rho,\mathbf{c},\mathbf{v}}(\mathbf{x}) = \mathbf{x} + s_{\rho,\mathbf{c},\mathbf{v}}(\mathbf{x})\mathbf{v}.$$

The implicit function theorem can be applied to prove that $s_{\rho,\mathbf{c},\mathbf{v}}$ is smooth, and as a consequence $\psi_{\rho,\mathbf{c},\mathbf{v}}$ is a smooth diffeomorphism. □

*With Gaussian Radial Basis Function (RBF) kernel:*

We now consider the following kernel (with $\rho \in \mathbb{R}_{>0}$):

$$k_\rho(\mathbf{x}, \mathbf{y}) = \exp\left(-\rho^2\|\mathbf{x} - \mathbf{y}\|^2\right).$$

We have:

$$\frac{\partial k_\rho}{\partial \mathbf{x}}(\mathbf{x}, \mathbf{y}) \cdot \mathbf{v} = -2\rho^2 \exp\left(-\rho^2\|\mathbf{x} - \mathbf{y}\|^2\right)(\mathbf{x} - \mathbf{y}) \cdot \mathbf{v},$$

with the lower bound:

$$\frac{\partial k_\rho}{\partial \mathbf{x}}(\mathbf{x}, \mathbf{y}) \cdot \mathbf{v} \geq -2\rho^2 \exp\left(-\rho^2\|\mathbf{x} - \mathbf{y}\|^2\right)\|\mathbf{x} - \mathbf{y}\| \cdot \|\mathbf{v}\|.$$

The expression on the right takes its minimum for $\|\mathbf{x} - \mathbf{y}\| = \frac{1}{\sqrt{2}\rho}$, which yields:

$$\frac{\partial k_\rho}{\partial \mathbf{x}}(\mathbf{x}, \mathbf{y}) \cdot \mathbf{v} \geq -\sqrt{2}\|\mathbf{v}\|\rho \exp\left(-\frac{1}{2}\right).$$

We pose $\rho_{\max}(\mathbf{v}) = \frac{1}{\sqrt{2}\|\mathbf{v}\|} \exp\left(\frac{1}{2}\right)$. Applying Theorem 1, $\mathbf{v} = 0$ or $\rho < \rho_{\max}(\mathbf{v})$ implies that $\psi_{\rho,\mathbf{c},\mathbf{v}}$ is a smooth diffeomorphism. In that case, $s_{\rho,\mathbf{c},\mathbf{v}}(\mathbf{x})$, and as a result $\psi^{-1}_{\rho,\mathbf{c},\mathbf{v}}(\mathbf{x})$, can be very efficiently computed with Newton's method.

## 3. A diffeomorphic matching algorithm

In this section we are interested in the following problem: given two sequences of distinct points $\mathbf{X} = (\mathbf{x}_i)_{i \in \{0,\ldots,N\}}$ and $\mathbf{Y} = (\mathbf{y}_i)_{i \in \{0,\ldots,N\}}$, compute a diffeomorphism $\Phi$ that maps each $\mathbf{x}_i$ onto $\mathbf{y}_i$, either exactly or approximately. More formally, defining $\mathrm{dist}(\mathbf{A}, \mathbf{B}) = \frac{1}{N+1}\sum_i \|\mathbf{a}_i - \mathbf{b}_i\|^2$ for two sequences $\mathbf{A}$ and $\mathbf{B}$ of $N + 1$ points, and denoting by $\Phi(\mathbf{X})$ the sequence of points $(\Phi(\mathbf{x}_i))_{i \in \{0,\ldots,N\}}$, we want to find a diffeomorphism $\Phi$ that minimizes $\mathrm{dist}(\Phi(\mathbf{X}), \mathbf{Y})$.

### 3.1. State of the art

Since the sequences $\mathbf{X}$ and $\mathbf{Y}$ can be very different in shape, to the best of our knowledge the state-of-the-art existing techniques to solve this problem are based on the Large Deformation Diffeomorphic Metric Mapping (LDDMM) framework introduced in the seminal article by Joshi and Miller [6]. Its core idea is to work with a time dependent vector field $v(\mathbf{x}, t) \in \mathbb{R}^d$ ($t \in [0, 1]$), and define a flow $\phi(\mathbf{x}, t)$ via the transport equation:

$$\frac{d\phi(\mathbf{x}, t)}{dt} = v(\phi(\mathbf{x}, t), t),$$

with $\phi(\mathbf{x}, 0) = \mathbf{x}$. With a few regularity conditions on $v$ (see [7] for specific requirements), $\mathbf{x} \mapsto \phi(\mathbf{x}, t)$ is a diffeomorphism. The resulting diffeomorphism $\Phi(\mathbf{x}) = \phi(\mathbf{x}, 1)$ is given by:

$$\Phi(\mathbf{x}) = \mathbf{x} + \int_0^1 v(\phi(\mathbf{x}, t), t)dt.$$

Using an appropriate Hilbert space, the vector fields $\mathbf{x} \mapsto v(\mathbf{x}, t)$ can be associated to an infinitesimal cost whose integration is interpreted as a deformation energy.

Various gradient descent algorithms have been proposed to optimize $v$ with respect to a cost that depends both on the deformation energy and on the accuracy of the mapping, whether the objective is to map curves [8], surfaces [9], or, as in our case, points [10].