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Multiple integral inequalities and stability analysis of time delay systems

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HIGHLIGHTS

- New multiple integral inequalities are derived.
- A set of sufficient LMI stability conditions for time delay systems are derived.
- The LMI conditions are arranged into a bidirectional hierarchy.

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ABSTRACT

This paper is devoted to stability analysis of continuous-time delay systems based on a set of Lyapunov–Krasovskii functionals. New multiple integral inequalities are derived that involve the famous Jensen's and Wirtinger's inequalities, as well as the recently presented Bessel–Legendre inequalities of Seuret and Gouaisbaut (2015) and the Wirtinger-based multiple-integral inequalities of Park et al. (2015) and Lee et al. (2015). The present paper aims at showing that the proposed set of sufficient stability conditions can be arranged into a bidirectional hierarchy of LMIs establishing a rigorous theoretical basis for comparison of conservatism of the investigated methods. Numerical examples illustrate the efficiency of the method.

1. Introduction

Time delays are present in many physical, industrial and engineering systems. The delays may cause instability or poor performance of systems, therefore much attention has been devoted to obtain tractable stability criteria of systems with time delay during the past few decades (see e.g. the monographs [1–3], some recent papers [4–17] and the references therein). Several approaches have been elaborated and successfully applied for the stability analysis of time delay systems (see the references above for excellent overviews).

Lyapunov method is one of the most fruitful fields in the stability analysis of time delay systems. On the one hand, more and more involved Lyapunov–Krasovskii functionals (LKF) have been introduced during the past decades. On the other hand, much effort has been devoted to derive more and more tight inequalities (Jensen's inequality and different forms of Wirtinger's inequality [1,2,4–14,18,19], etc.) for the estimation of quadratic single, double and multiple integral terms in the derivative of the LKF. Simultaneously, augmented state vectors are introduced in part as a consequence of the improved estimations, in part on an ad hoc basis. The effectiveness of different methods is mainly compared using some numerical examples. Recently, the authors of [4,13] have introduced a very appealing idea of the hierarchy of LMI conditions offering a rigorous theoretical basis for comparison of stability LMI conditions. Based on Legendre polynomials, they proposed a generic set of single integral inequalities opening the way to the derivation of a set of stability conditions forming a hierarchy of LMIs. A further possibility for the derivation of improved stability conditions have been proposed by [5,6] using multiple integral quadratic terms in the LKF, together with Wirtinger-based multiple integral inequalities. Naturally the question arises: how these two lines of investigations are related to each other, and how sufficient stability conditions can be derived unifying the approaches of using multiple integral quadratic terms in the LKF and refined estimations of these integral terms.

The aim of the present work is to answer these questions. On the one hand, multiple integral inequalities based on orthogonal hypergeometric polynomials will be derived that extend the results

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of [4,13] *to multiple integrals and improve the estimations of* [5,6]. *On the other hand, a multi-parametric set of LMI conditions will be constructed, and it will be shown that a two parametric subset forms a bidirectional hierarchy of LMIs.*

Analogous results have been presented for discrete-time systems in [20].

The paper is organized as follows. In Section 2 it is shown, how the quadratic terms of the derivative of the LKF can be estimated by Bessel-type inequalities. It is also proven that these estimations relevantly improve a recently published result. A sufficient condition of asymptotic stability is presented in the form of an LMI in Section 3. The hierarchy of LMI conditions is established then in Section 4. Some benchmark numerical examples are shown in Section 5, the results of which are compared to earlier ones known from the literature. Finally, the conclusions will be drawn.

The notations applied in the paper are very standard, therefore we mention only a few of them. Symbol $A \otimes B$ denotes the Kronecker-product of matrices A, B, while $\mathbf{S}_n$ and $\mathbf{S}_n^+$ are the set of symmetric and positive definite symmetric matrices of size $n \times n$, respectively.

2. Multiple integral inequalities

2.1. Preliminaries

The paper deals with the stability analysis of the following continuous-time delay system

$$\dot{x}(t) = Ax(t) + A_{d_1}x(t-\tau) + A_{d_2}\int_{t-\tau}^{t} x(s)\,\mathrm{d}s, \quad t \geq 0, \tag{1}$$

$$x_0(t) = \varphi(t), \quad t \in [-\tau, 0], \tag{2}$$

where $x(t) \in \mathbf{R}^{n_x}$ is the state, A, A_{d_1} and A_{d_2} are given constant matrices of appropriate size, the time delay τ is a known positive constant and $x_0(\cdot)$ is the initial function.

(A.) *A Bessel-type inequality.* Let $\mathbf{E}$ be a Euclidean space with the scalar product $\langle\cdot,\cdot\rangle$, and let $\pi_j \in \mathbf{E}$, $(j = 0, 1, \ldots)$ form an orthogonal system. Let $\overline{n} \geq 1$ be a given integer. For any $f, g \in \mathbf{E}^{\overline{n}}$, define $\langle f, g\rangle = \sum_{i=1}^{\overline{n}}\langle f_i, g_i\rangle$. Let $W \in \mathbf{S}_{\overline{n}}^+$. For any $f \in \mathbf{E}^{\overline{n}}$, consider the functional

$$J_W(f) = \langle f, Wf\rangle. \tag{3}$$

Lemma 1. *If $\nu \geq 0$ is a given integer, then the following inequality holds*

$$J_W(f) \geq \sum_{j=0}^{\nu}\frac{1}{\|\pi_j\|^2}w_j^T W w_j, \tag{4}$$

where $w_j = \langle f, \pi_j\rangle$, and the scalar product is taken componentwise.

Proof. The proof is standard, therefore it is omitted.

(B.) *Orthogonal hypergeometric polynomials.* Suppose that $\ell \geq 0$ is a given integer and consider the closed interval $[a, b]$. For functions $g_1, g_2 \in L_2[a, b]$ define a scalar product by

$$\langle g_1, g_2\rangle_{\ell,[a,b]} = \int_a^b \left(\frac{s-a}{b-a}\right)^{\ell} g_1(s)g_2(s)\,\mathrm{d}s. \tag{5}$$

It is easy to see that $\langle g_1, g_2\rangle_{\ell,[a,b]}$ can equivalently be expressed as

$$\langle g_1, g_2\rangle_{\ell,[a,b]} = \frac{\ell!}{(b-a)^{\ell}}\int_a^b\int_{v_1}^b\cdots\int_{v_\ell}^b g_1(s)g_2(s)\,\mathrm{d}s\,\mathrm{d}v_\ell\ldots\mathrm{d}v_1, \quad \text{if } \ell > 0. \tag{6}$$

(If $\ell = 0$, then a single integral is considered.) Substitute $s \in [a, b]$ by $s = a + (b-a)x$, where $x \in [0, 1]$, and set $G_i(x) = g_i(a + (b-a)x)$, $(i = 1, 2)$ on the right hand side of (5), then we obtain that

$$\begin{aligned}\langle g_1, g_2\rangle_{\ell,[a,b]} &= (b-a)\int_0^1 x^{\ell}G_1(x)G_2(x)\,\mathrm{d}x \\ &= (b-a)\langle G_1, G_2\rangle_{\ell,[0,1]}.\end{aligned} \tag{7}$$

Thus it is sufficient to consider the orthogonal polynomials with respect to $\langle\cdot,\cdot\rangle_{\ell,[0,1]}$.

For any fixed non-negative integer ℓ, let us denote by $P_{\ell,n}$, $(n = 0, 1, \ldots)$ the polynomials of degree n orthogonal with respect to $\langle\cdot,\cdot\rangle_{\ell,[0,1]}$. (For general theory see e.g. [21].) They can be given by the two parameters generalization of the Rodrigues-formula:

$$P_{\ell,0}(x) \equiv 1, \tag{8}$$

$$P_{\ell,n}(x) = \frac{1}{n!}\frac{1}{x^{\ell}}\frac{d^n}{\mathrm{d}x^n}\left(x^{\ell}(x^2-x)^n\right), \quad n = 1, 2, \ldots \tag{9}$$

For $\ell = 0$, this is the usual Rodrigues formula for the shifted Legendre polynomials.

We note that polynomials (8)–(9) satisfy certain hypergeometric-type differential equation (see e.g. [22,23]). This is why they are frequently called "orthogonal hypergeometric polynomials". By straightforward calculation, it can be shown that they have the properties

$$\text{(i) } P_{\ell,n}(x) = (-1)^n\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{\ell+k+n}{\ell+k}x^k, \quad \ell, n \geq 0, \tag{10}$$

$$\text{(ii) } \left\|P_{\ell,n}\right\|^2_{\ell,[0,1]} = \int_0^1 x^{\ell}P^2_{\ell,n}(x)\,\mathrm{d}x = \frac{1}{\ell+2n+1}, \tag{11}$$

$$\text{(iii) } P_{\ell,n}(0) = (-1)^n\frac{\ell+n}{n}, \quad P_{\ell,n}(1) = 1. \tag{12}$$

The polynomials

$$p_{\ell,n}(t) = P_{\ell,n}\left(\frac{t-a}{b-a}\right) \tag{13}$$

are orthogonal with respect to the scalar product (5), and

$$\left\|p_{\ell,n}\right\|^2_{\ell,[a,b]} = \frac{b-a}{\ell+2n+1}, \quad p_{\ell,n}(a) = (-1)^n\frac{\ell+n}{n}, \ p_{\ell,n}(b) = 1. \tag{14}$$

In what follows, polynomials $q_{\ell,\ell+j}(x) = x^{\ell}P_{\ell,j}(x)$ (for $\ell > 0$, $j = 0, 1, \ldots$) and $\frac{\mathrm{d}}{\mathrm{d}x}q_{\ell,\ell+j}(x)$ (for $\ell \geq 0$, $j = 0, 1, \ldots$) have to be expressed in terms of the shifted Legendre polynomials $P_{0,0}$, $P_{0,1}, \ldots$. To this end, consider the nonnegative integers $n_1 \leq n_2, K$, and introduce the notations

$$X_{n_1,n_2} = \left(x^{n_1}, x^{n_1+1}, \ldots, x^{n_2}\right)^T,$$
$$\Pi_{\ell,K}(x) = \left(P_{\ell,0}(x), P_{\ell,1}(x), \ldots, P_{\ell,K}(x)\right)^T,$$
$$D_{n_1,n_2} = \operatorname{diag}\{n_1, n_1+1, \ldots, n_2\},$$

and $G(\ell, K) \in \mathbf{R}^{(K+1)\times(K+1)}$ with elements $G(\ell, K)_{1,1} = 1$, $G(\ell, K)_{l,k} = 0$, if $1 \leq l < k \leq K+1$,

$$G(\ell, K)_{l+1,k+1} = (-1)^{l+k}\prod_{j=0}^{k-1}\frac{l-j}{k-j}\prod_{i=1}^{l}\frac{\ell+k+i}{i}, \quad \text{if } l = 1, \ldots K, \ k = 0, \ldots, l,$$

(The void product equals to 1 by definition.)