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Switched systems with multiple invariant sets ☆

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HIGHLIGHTS

- Under dwell time constraint, switched systems converge to a set in finite time.
- Subsequently, trajectories remain within a larger invariant set.
- Optimization of tuning parameters is a tradeoff between spatial/ temporal bounds.

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ABSTRACT

This paper explores dwell time constraints on switched systems with multiple, possibly disparate invariant limit sets. We show that, under suitable conditions, trajectories globally converge to a superset of the limit sets and then remain in a second, larger superset. We show the effectiveness of the dwell-time conditions by using examples of switching limit cycles commonly found in robotic locomotion and flapping flight.

1. Introduction

Bifurcations have been of interest to dynamical systems theory for decades. However, most control strategies view such behavior as damaging and try to mitigate it [1]. Relatively less work actively inserts bifurcations as part of a control strategy. One example is using a classic Hopf bifurcation for mode-switching between flapping and gliding flight in micro-aerial vehicles [2]. The authors consider a supercritical Andronov–Hopf bifurcation model of $\mathbf{x} = (u; v)$:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t; \rho) = \begin{pmatrix} -\lambda/\rho^2 \left(u^2 + v^2 - \rho^2\sigma\right) u - \omega(t)v \\ \omega(t)u - \lambda/\rho^2 \left(u^2 + v^2 - \rho^2\sigma\right) v \end{pmatrix} \tag{1}$$

with $\sigma = 1$. For a positive rate of convergence $\lambda > 0$, it can be easily shown that any initial trajectory $(u; v) \neq \mathbf{0}$ exponentially converges to a circle of the radius ρ rotating at the time-varying frequency $\omega(t)$ with bounded $\dot{\omega}(t)$. If $\sigma \leq 0$, bifurcation occurs and the system globally converges to the origin, which is useful for fast inhibition of oscillation. Fast inhibition and synchronization of oscillators are key properties for many neurobiologically-inspired control schemes.

Another possible application is walking robots. Fig. 1 shows a hypothetical switching pattern for a walking robot application utilizing central pattern generation. A guidance/navigation engineer may design limit cycle subsystems for walking and jumping modes (shown as a Hopf oscillator and a Van der Pol oscillator), while utilizing steady-state control strategies for static balancing or tasks requiring fine motor control.

Mode-switching also implicates a large body of literature on switched systems [3]. Most work on stability of switched systems assumes that all subsystems have a common equilibrium point. [4–6] consider weak Lyapunov functions in the style of LaSalle for a common equilibrium. [7] considers equilibrium location changes, but holds the vector field constant. They connect the result to averaging theory. [8] considers practical stability of affine systems with multiple distinct equilibria. Alpcan and Başar investigated dwell time criteria for nonlinear globally exponentially stable subsystems which could have differing equilibria [9]. Such systems have no single globally attractive equilibrium point. The authors of [9] reported an explicit construction of the dwell time and a

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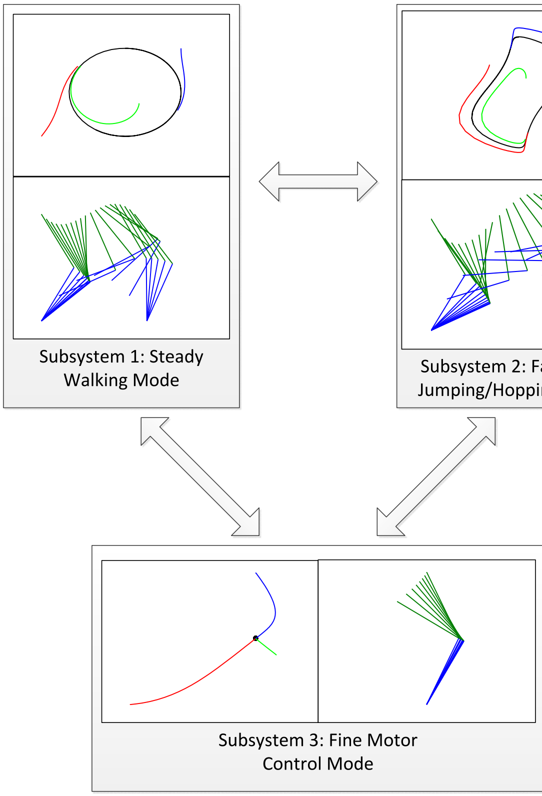

Fig. 1. Schematic of mode switching with non-equilibrium limit sets.

conservative invariant set. This paper is inspired by that work and is a generalization of it. We generalize their result to switched systems where each subsystem may have multiple invariant sets. We pursue a similar dwell time strategy in order to provide spatial bounds for the switched system.

Systems with bifurcation often contain multiple ω-limit sets which cannot be globally exponentially stable. Instead, results such as LaSalle's invariant set theorem [10] allow us to analyze asymptotic stability of this larger class of systems. LaSalle's theorem and much of the switched systems literature are both Lyapunov-based, and we will make use of Lyapunov functions to define all the relevant sets.

Section 2 provides background assumptions and definitions. Section 3 begins by reconsidering existing results. Section 3.3 through 3.5 present two methods to accomplish the goal. Choice of a particular method will depend on specific situations and design constraints. Section 4 shows two numerical examples, and Section 5 provides concluding comments.

2. Preliminaries and definitions

Consider a set of continuous-time dynamical systems defined by

$$\dot{\mathbf{x}} = \mathbf{f}_p(\mathbf{x}), \tag{2}$$

where $\mathbf{x} \in \mathbb{R}^n$ and $p \in \mathcal{P}$, with some index set $\mathcal{P} = \{p_1, p_2, \ldots, p_{\max}\}$. A piecewise constant switching signal $\sigma : [0, \infty) \to \mathcal{P}$ specifies the active subsystem at each time. Assume, for ease of analysis, that $\mathbf{f}_p$ are each continuous with continuous first partials. Together, (2), the index set, and the switching signal define a switched system.

We will consider a constraint on how quickly the switching signal can make consecutive switches.

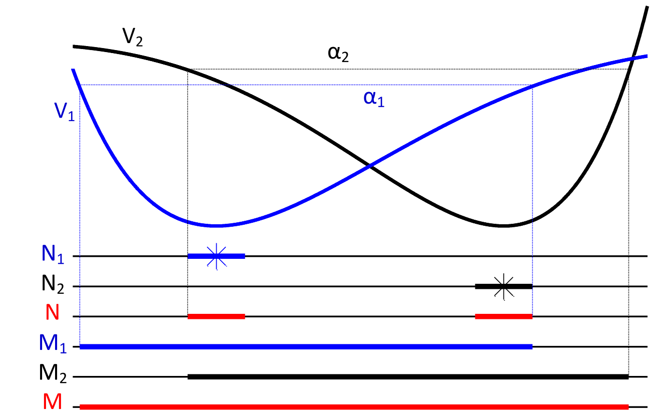

Fig. 2. Qualitative example of how $\mathcal{N}$ and $\mathcal{M}$ are built for a switched system consisting of two subsystems, each with a single equilibrium, but at different locations.

Definition. Consider a switched system with switching times $\{t_1, t_2, \ldots\}$. It is said to have *dwell time* τ if $t_{i+1} - t_i \geq \tau \quad \forall i \in \mathbb{N}$.

In this paper, integer subscripts on t are reserved for switching times. Denote the limit from the right/left as superscript $+/-$, respectively.

Next, we review and introduce some important subsets of $\mathbb{R}^n$. Assume that each subsystem has a (possibly different) $\mathcal{C}^1$ Lyapunov-like function, which is bounded above and below on every bounded subset of $\mathbb{R}^n$. Furthermore, assume that each is radially unbounded ($V_p(\mathbf{x}) \to \infty$ as $\|\mathbf{x}\| \to \infty$). This ensures that every sublevel set describes a compact region. We assume for the remainder of this paper that the minimum value of each V_p is zero. Define

$$\mathcal{G}_p = \left\{\mathbf{x} \in \mathbb{R}^n | V_p(\mathbf{x}) = 0\right\} \tag{3}$$

as the set which attains the minimum value of V_p. Let κ be a positive constant and define

$$\mathcal{N}_p(\kappa) = \left\{\mathbf{x} \in \mathbb{R}^n | V_p(\mathbf{x}) \leq \kappa\right\}, \tag{4}$$

a closed κ-neighborhood of $\mathcal{G}_p$. For the purposes of Theorem 1, $\mathcal{N}_p(\kappa)$ is connected, but it is not necessarily connected in the remainder of the paper (see Fig. 4). Let

$$\mathcal{N}(\kappa) = \bigcup_{p \in \mathcal{P}} \mathcal{N}_p(\kappa). \tag{5}$$

Additionally, define a superset, $\mathcal{M}(\kappa)$, in a series of steps with

$$\alpha_p(\kappa) = \max_{\mathbf{x} \in \mathcal{N}(\kappa)} V_p(\mathbf{x}), \tag{6}$$

and

$$\mathcal{M}_p(\kappa) = \left\{\mathbf{x} \in \mathbb{R}^n : V_p(\mathbf{x}) \leq \alpha_p(\kappa)\right\}. \tag{7}$$

Finally, we create a closed union of closed sublevel sets,

$$\mathcal{M}(\kappa) = \bigcup_{p \in \mathcal{P}} \mathcal{M}_p. \tag{8}$$

Notice that the dependence on κ carries through once we use it in $\mathcal{N}(\kappa)$. For the purposes of Theorem 1, $\mathcal{M}$ is a connected superset of $\mathcal{N}$. Theorem 4 will introduce a different notion which is not necessarily connected. Fig. 2 provides a one-dimensional example to help visualize these sets.

3. Stability results

We first restructure the result in [9] to add clarity and to better facilitate the generalization presented in this paper.