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Synchronization of diffusively coupled systems on compact Riemannian manifolds in the presence of drift

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ABSTRACT

Recently, it has been shown that the synchronization manifold is an asymptotically stable invariant set of diffusively coupled systems on Riemannian manifolds. We regionally investigate the stability properties of the synchronization manifold when the systems are subject to drift. When the drift vector field is QUAD (i.e. satisfies a certain quadratic inequality) and the underlying Riemannian manifold is compact, we prove that a sufficiently large algebraic connectivity of the underlying graph is sufficient for the synchronization manifold to remain asymptotically stable. For drift vector fields which are QUAD or contracting, we explicitly characterize the rate at which the solution converges to the synchronization manifold. Our main result is that the synchronization manifold is asymptotically stable even for drift vector fields which are only locally Lipschitz continuous, as long as the algebraic connectivity of the underlying graph is sufficiently large.

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1. Introduction

Lately, significant effort has been dedicated to the study of synchronizing systems that live on manifolds [1–5]. Scardovi et al. have studied synchronizing systems on tori [1] and Sarlette et al. have focused on synchronizing systems on the orthogonal group [2]. A quite general study of synchronization on manifolds has been conducted by Sarlette and Sepulchre [3]. Therein, the authors embed the manifold under investigation into Euclidean space and analyze the convergence behavior of the systems from an extrinsic (i.e. projective) point of view. A very general review paper regarding the study of synchronization on manifolds was published by Sepulchre [4]. Tron et al. [5] studied synchronization from an intrinsic point of view whilst characterizing the region of attraction of the synchronization manifold precisely, therein treating systems that evolve in discrete time. The work of Sarlette and Sepulchre [3] as well as the work of Tron et al. [5] are in the same spirit as the present paper, but we will take the intrinsic point of view (i.e. we will not embed the manifold under investigation) and consider systems evolving in continuous time.

Contribution. While Sarlette and Sepulchre [3] and Tron et al. [5] proved asymptotic stability of the synchronization manifold for diffusively coupled systems without drift, making the synchronization manifold a submanifold of equilibria, we regionally investigate the stability properties of the synchronization manifold for diffusively coupled systems that are subject to drift, allowing for nontrivial solutions on the synchronization manifold. This is of interest when studying synchronization of oscillators (admitting periodic orbits on the synchronization manifold [6]) or cooperative tasks (in which a prespecified trajectory must be obeyed on the synchronization manifold [7]). The according drift vector fields can appear nontrivial when studying them in ambient space, but often look more natural when studying them on the manifold whose product is an invariant set of the coupled systems. We show that the synchronization manifold is asymptotically stable if the drift vector field satisfies certain smoothness assumptions (e.g. local Lipschitz continuity) and explicitly characterize the rate at which the solution converges to the synchronization manifold. The presented approach is related to the approach of DeLellis et al. [8], who have studied synchronization of diffusively coupled systems whose drift vector fields are QUAD, contracting, or globally Lipschitz continuous in Euclidean space, but extends the results to Riemannian manifolds. We illustrate our findings on the special orthogonal group with application to the synchronization of rigid bodies.

Structure of the paper. The remainder of the paper is structured as follows; in Section 2, we restate the results of Tron et al. [5] in continuous time. In Section 3, we regionally analyze the stability properties of the synchronization manifold when the diffusively coupled systems are subject to drift. Therein, we consider QUAD

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drift vector fields in Section 3.1, contracting drift vector fields in Section 3.2, and locally Lipschitz continuous drift vector fields in Section 3.3. We illustrate all utilized concepts and notions on the example of the special orthogonal group in Section 4 and conclude the paper with Section 5. We introduce basic notions and preliminary results from Riemannian geometry in the Appendix.

2. Synchronization without drift

In this section, we review the results of Tron et al. [5] in a continuous-time setting. Let $\mathscr{M}$ be a Riemannian manifold and $d : \mathscr{M} \times \mathscr{M} \to \mathbb{R}$ the distance function induced by the length of minimizing geodesics. Define the nonnegative function $P : \mathscr{M}^N \to \mathbb{R}$,

$$P(x) = \sum_{i=1}^{N}\sum_{j=1}^{N} \frac{w_{ij}}{4} d\left(x_i, x_j\right)^2, \tag{1}$$

where for all $i, j \in \{1 \cdots N\}$, $w_{ij} \geq 0$ and $w_{ij} = w_{ji}$. We associate an undirected, weighted graph $\mathscr{G}$ with N vertices to the nonnegative scalars w_{ij} by letting w_{ij} be the weight of the arc connecting vertex i and vertex j. An undirected, weighted graph is a triple $(\mathscr{V}, \mathscr{A}, \mathscr{W})$, where $\mathscr{V}$ is the set of vertices (here: $\mathscr{V} = \{1 \cdots N\}$), $\mathscr{A} \subset \mathscr{V} \times \mathscr{V}$ are the arcs (here: $\mathscr{A} = \{(i, j) \in \{1 \cdots N\}^2 | w_{ij} > 0\}$), and the positive function $\mathscr{W} : \mathscr{A} \to \mathbb{R}$ are the weights (here $\mathscr{W} : (i, j) \mapsto w_{ij}$). Note that for us, the weights w_{ij} implicitly define both $\mathscr{A}$ as well as $\mathscr{W}$, i.e. $\mathscr{G}$ is uniquely determined by w_{ij} and vice versa. A path is a sequence of elements of $\mathscr{V}$ for which each consecutive pair of vertices is connected by an arc. The graph $\mathscr{G}$ is said to be connected, if for every two members of $\mathscr{V}$, there exists a path such that one member is the start vertex and the other member is the end vertex. It is a fundamental result of algebraic graph theory [9,10] that, if $\mathscr{G}$ is connected, the Laplacian L of $\mathscr{G}$ given by $-w_{ij}$ being the ith element of the jth column and $\sum_{j=1, j\neq i}^{N} w_{ij}$ being the ith diagonal element, is positive semidefinite with nullspace $\Omega = \{y \in \mathbb{R}^N | \exists \alpha \in \mathbb{R} : y = 1_N \alpha\}$, where 1_N is the N-fold vector of ones, and that the form $y^\top L y$ is positive definite for all vectors y from the orthogonal complement of Ω. The latter implies the existence of a positive scalar λ such that, for all y from the orthogonal complement of Ω, it is true that $y^\top L y \geq \lambda y^\top y$. This quantity λ is called the algebraic connectivity of $\mathscr{G}$ and λ can be increased arbitrarily by appropriate choice of weights w_{ij}, i.e. whenever it is desirable to increase λ, this can be done by appropriately increasing certain weights.

Let $x_i \in \mathscr{M}$ denote the ith component of $x \in \mathscr{M}^N$. The function P attains the value zero at the so-called synchronization manifold

$$S = \{x \in \mathscr{M}^N | \exists y \in \mathscr{M} : \forall i \in \{1 \cdots N\}, x_i = y\}. \tag{2}$$

Substitution verifies that $P(S) = 0$ (i.e. for all $x \in S$, $P(x) = 0$).

Lemma 1 (*[5, Lemma 4]*). *Let $\mathscr{G}$ be connected. Then $P(x) = 0$ if and only if $x \in S$.*

Let us characterize an open neighborhood

$$U_S^r = \{x \in \mathscr{M}^N | \exists y \in \mathscr{M} : \forall i \in \{1 \cdots N\} : d(y, x_i) < r\} \tag{3}$$

of S. For convergence characterizations, it is of interest to find a radius r such that S is the set of all critical points of P in U_S^r. In the light of Lemma A.7 (cf. Appendix), the lower bound

$$r^\star = \frac{1}{2}\min\left\{\operatorname{inj}\mathscr{M}, \frac{\pi}{\sqrt{\Delta}}\right\} \tag{4}$$

on the convexity radius of $\mathscr{M}$, where $\operatorname{inj}\mathscr{M}$ denotes the injectivity radius of $\mathscr{M}$ and Δ is an upper bound for the sectional curvature of $\mathscr{M}$ (with the convention that $\Delta \leq 0$ implies $\frac{1}{\sqrt{\Delta}} = \infty$), is a candidate for such a radius [11, Section 6.5].

Theorem 1 (*[5, Theorem 5]*). *Let $\mathscr{G}$ be connected and $r^\star$ denote the lower bound on the convexity radius of $\mathscr{M}$. Then a point $x \in U_S^{r^\star}$ is a critical point of P if and only if $x \in S$.*

If we declare the system

$$\dot{x} = -\operatorname{grad} P(x) =: X(x), \tag{5}$$

then S is an invariant set of (5) and we say that the N components x_i of x are diffusively coupled. Let $\phi_x : \mathscr{M}^N \times (-\epsilon, \epsilon) \to \mathscr{M}^N$, $(x_0, t) \mapsto \varphi_x(x_0, t)$ denote the solution to (5), where $(-\epsilon, \epsilon)$ is an interval of existence. If

$$\lim_{t\to\infty} d_N\left(\phi_x(x_0, t), S\right) = 0, \tag{6}$$

then we say that the N components x_i of x synchronize. In the following, let U_P^α denote

$$U_P^\alpha = \{x \in \mathscr{M}^N | P(x) \leq \alpha\}. \tag{7}$$

Proposition 1 (*along the lines of [5]*). *Let $\mathscr{G}$ be connected and $r^\star$ denote the lower bound on the convexity radius of $\mathscr{M}$. If $\mathscr{M}$ is compact, then S is an asymptotically stable invariant set of (5). Moreover, for all α such that $U_P^\alpha \subset U_S^{r^\star}$, U_P^α is a subset of the region of asymptotic stability of S.*

Proof. For all $x \in \mathscr{M}^N$, $P(x) \geq 0$. Due to Lemma 1, $P(x) = 0$ if and only if $x \in S$. Thus, S is a critical submanifold of P. Hence, S is an invariant set of (5). As $\mathscr{M}$ is compact, $\mathscr{M}^N$ is compact and since S is a closed subset of $\mathscr{M}^N$, S is compact. P maps $U_S^{r^\star}$ to $\mathbb{R}$ and by Lemma 1, $P\left(U_S^{r^\star} \setminus S\right) > 0$ as well as $P(S) = 0$ hold true. The Lie derivative of P along X is given by

$$\mathrm{L}_X P(x) = -\left\langle \operatorname{grad} P(x), \operatorname{grad} P(x)\right\rangle. \tag{8}$$

By virtue of Theorem 1, $\mathrm{L}_X P\left(U_S^{r^\star} \setminus S\right) < 0$ and $\mathrm{L}_X P(S) = 0$. It follows from the invariance of sublevel sets of P together with Lyapunov's Direct Method [12, Theorem 2.2] and LaSalle's Invariance Principle [13] that S is asymptotically stable and for all α such that $U_P^\alpha \subset U_S^{r^\star}$, U_P^α is a subset of the region of asymptotic stability of S. This was the statement of the proposition. □

3. Synchronization with drift

In the previous section, we have studied (5), i.e. diffusively coupled systems on Riemannian manifolds that are not subject to drift. We now investigate the case where each component x_i of x is subject to a drift vector field f, i.e. f is a vector field on $\mathscr{M}$ (for every $x_i \in \mathscr{M}$, f maps x_i to an element of $\mathrm{T}_{x_i}\mathscr{M}$). Then (5) reads

$$\dot{x} = F(x) - \operatorname{grad} P(x) =: X(x), \tag{9}$$

where

$$F(x) = \left(f(x_1) \cdots f(x_N)\right). \tag{10}$$

As for all $x \in S$, $F(x) \in \mathrm{T}_x S$ and $\operatorname{grad} P(x) = 0$, S is an invariant set of (9). If, in contrast, the drift vector fields governing the individual systems are nonidentical, then S is no invariant set of (9) and can thus be no asymptotically stable invariant set of (9).

We are concerned with properties of f that are sufficient to let S be an asymptotically stable invariant set of (9). In doing so, we admit for drift vector fields whose integral curves can have arbitrary complexity. This brings one into the position to study complex vector fields (such as those of oscillators) in a more natural fashion then studying them in ambient space.

In the following, with $\bar{x}$, we will refer to the Riemannian center of mass of x, i.e. the minimizer of $\sum_{i=1}^{N} d(x_i, y)^2$ in y, whenever it exists and is unique. We will employ $\bar{x}$ to measure the distance of x to S with the error function

$$e(x) = \frac{1}{2} d_N\left(x, (\bar{x} \cdots \bar{x})\right)^2 \tag{11}$$

whenever $\bar{x}$ exists and is unique. By U_e^α, we denote

$$U_e^\alpha = \{x \in \mathscr{M}^N | e(x) \leq \alpha\}. \tag{12}$$

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