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# Positive state controllability of positive linear systems

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ABSTRACT

Controllability of positive systems by positive inputs arises naturally in applications where both external and internal variables must remain positive for all time. In many applications, particularly in population biology, the need for positive inputs is often overly restrictive. Relaxing this requirement, the notion of positive state controllability of positive systems is introduced. A connection between positive state controllability and positive input controllability of a related system is established and used to obtain Kalman-like controllability criteria. In doing so we aim to encourage further study in this underdeveloped area.


## 1. Introduction

Controllability is one of the most fundamental concepts in control theory. For finite-dimensional, linear, time-invariant, continuous time systems the notions of reachability, controllability and null controllability are all equivalent. The formulation of these concepts dates back to Kalman [1] and, as is well-known, their appeal lies in the interplay between analytic and algebraic concepts. For instance, the existence of a control steering the system to a desired state is equivalent to the reachability matrix having full rank.

Controllability does not *a priori* respect any (componentwise) nonnegativity of a system. This is problematic for many physically motivated applications, where state and input variables correspond to quantities that cannot take negative values. The need for nonnegative variables motivated the development of positive system theory and there now exist several textbooks on the subject (for example, [2–4]). Naturally, controllability, that is *positive input controllability*, in such a framework is more limited than the general case, but the situation is well understood ([5,6] and the references therein). A key feature of positive system theory is the notion that both the state and the input variables must be nonnegative.

It is of interest, however, to consider

$$x(t+1) = Ax(t) + Bu(t), \quad t \in \mathbb{N}_0, \tag{1.1}$$

where $A$ and $B$ are componentwise nonnegative under the constraints that just the state must be nonnegative—what might be termed *positive state controllability*. There are conceivably many applications of such a framework, for example, in economic or logistic type models (see, for example, Miller and Blair [7]). Our primary example of where such a framework is necessary, however, is population ecology. Here matrix models are often used (see, for example, Caswell [8] or Cushing [9]) with the nonnegative state $x$ denoting a stage- or age-structured population, and the control $u$ denoting a conservation strategy or a form of pest control or harvesting. There are many papers (including, for example, [10–13]) where the model (1.1) is suitable for describing the addition or removal of individuals from a population and for a full description of these actions we require that $u$ can take negative values.

The framework of positive state controllability places a nonnegativity constraint on the codomain, and not on the domain, of the input-to-state map and it is not immediately clear that the positive input controllability theory is applicable. Here we demonstrate that under certain assumptions (reasonable for applications to population ecology) the problem of positive state controllability is equivalent to positive input controllability of a related positive system. Using this approach we characterise both the set of reachable states and the set of null controllable states of the pair $(A, B)$ under the constraint that the state must remain nonnegative. We demonstrate that, for example, the class of Leslie matrices [14] (with suitable control) that is frequently used in ecological modelling is positive state controllable, but often with negative control signals. We believe that there is seemingly a non-trivial 'middle ground'

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between the controllability of linear systems and the positive input controllability of positive systems that is worthy of in-depth study.

## 2. Positive state control

For $n \in \mathbb{N}$, $\mathbb{R}^n_+$ denotes the nonnegative orthant in $\mathbb{R}^n$ and $e_i \in \mathbb{R}^n$ is the $i$th standard basis vector. For vectors $x$ and matrices $X$, $x \geq 0$ (also $0 \leq x$) and $X \geq 0$ (also $0 \leq X$) denotes componentwise nonnegativity. The superscript $^T$ denotes matrix transposition. We are interested in the pair $(A, B)$ generating the controlled system (1.1) where $A, B \geq 0$ and the state $x$ is nonnegative.

Our main result is Theorem 2.6 which relates nonnegative state trajectories with possibly nonpositive inputs to nonnegative state trajectories with nonnegative inputs of a related system. Such a connection allows us to appeal to existing positive input control results for this related system. Our key assumption is as follows.

(A) Given the pair $(A, B) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m}$ with $A, B \geq 0$ there exists $F \in \mathbb{R}^{m\times n}$ such that with $\tilde{A} := A - BF$ both $\tilde{A} \geq 0$ and if $v \in \mathbb{R}^n_+$, $w \in \mathbb{R}^m$ satisfy $\tilde{A}v + Bw \geq 0$ then $w \geq 0$.

The idea of assumption (A) is that by decomposing $A$ into $\tilde{A} + BF$, negative controls $u$ in $Ax + Bu$ can be absorbed as $\tilde{A}x + B(Fx + u)$. Lemma 2.1 below gives a constructive characterisation of assumption (A) and demonstrates that if (A) holds then it holds for precisely one $F$ which can be calculated explicitly.

**Lemma 2.1.** *Assumption* (A) *holds for* $(A, B) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m}$ *with* $A, B \geq 0$ *if, and only if, there exist* $m$ *rows of* $B$ *such that the* $m \times m$ *submatrix, denoted as* $\underline{B}$*, formed by taking these* $m$ *rows from* $B$ *is a positive monomial matrix and*

$$A - B\underline{B}^{-1}\underline{A} \geq 0. \tag{2.1}$$

*Here* $\underline{A}$ *is formed of the* $m$ *rows of* $A$ *that appear in* $\underline{B}$*. Consequently,* (A) *holds if, and only if, it holds with* $F = \underline{B}^{-1}\underline{A}$ *so that* $\tilde{A} := A - BF \geq 0$.

To prove Lemma 2.1 we first need an intermediate result.

**Lemma 2.2.** *Given a pair* $(A, B)$ *with* $A, B \geq 0$ *assumption* (A) *holds for* $(A, B)$ *if, and only if, there exists* $F, H \in \mathbb{R}^{m\times n}$ *such that the following four conditions hold*

$$\begin{aligned} &(a)\ F, H \geq 0, && (b)\ A - BF \geq 0, \\ &(c)\ HA = F, && (d)\ HB = I_m. \end{aligned} \tag{2.2}$$

*Here* $I_p$ *with* $p \in \mathbb{N}$ *denotes the* $p \times p$ *identity matrix.*

**Proof.** Assumption (A) can be written as, there exists $F \in \mathbb{R}^{m\times n}$, $F \geq 0$ such that $\tilde{A} = A - BF \geq 0$ and for all $v \in \mathbb{R}^n$, $w \in \mathbb{R}^m$

$$\begin{bmatrix} \tilde{A} & B \\ I_n & 0 \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} \geq 0 \Rightarrow w = \begin{bmatrix} 0 & I_m \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} \geq 0. \tag{2.3}$$

By [15], (2.3) is equivalent to the existence of $\tilde{H} \in \mathbb{R}^{m\times(n+m)}$, $\tilde{H} \geq 0$ such that

$$\begin{aligned} &\begin{bmatrix} \tilde{H}_1 & \tilde{H}_2 \end{bmatrix} \begin{bmatrix} \tilde{A} & B \\ I_n & 0 \end{bmatrix} = \begin{bmatrix} 0 & I_m \end{bmatrix} \iff \\ &\tilde{H}_1\tilde{A} + \tilde{H}_2 = 0, \quad \text{and} \quad \tilde{H}_1 B = I_m. \end{aligned} \tag{2.4}$$

Since we require that $\tilde{H} \geq 0$ we see immediately from (2.4) that we can always take $\tilde{H}_2 = 0$ and hence by writing $H = \tilde{H}_1$ it follows that (A) is equivalent to the existence of $F, H \in \mathbb{R}^{m\times n}$ such that

$$F, H \geq 0, \qquad A - BF \geq 0, \qquad H\tilde{A} = 0, \qquad HB = I_m. \tag{2.5}$$

Using the fact that $\tilde{A} = A - BF$ so that $H\tilde{A} = HA - HBF$ we have that (2.5) is equivalent to (2.2), as required. □

**Proof of Lemma 2.1.** First assume that $B$ contains an $m \times m$ positive monomial submatrix, which we denote by $\underline{B}$. Let $\underline{A}$ denote the $m \times n$ submatrix of $A$ formed by taking the $m$ rows $\{i_1, \ldots, i_m\}$ of $A$ that appear in $\underline{B}$. Since $\underline{B}$ is positive monomial, it follows (from, for example, p. [16, p. 68]) that $\underline{B}$ has a positive inverse. Define

$$F := \underline{B}^{-1}\underline{A} \geq 0, \tag{2.6}$$

the nonnegativity following as $\underline{B}^{-1}, \underline{A} \geq 0$. Furthermore, by assumption $A - B\underline{B}^{-1}\underline{A} \geq 0$ and hence

$$\tilde{A} := A - BF = A - B\underline{B}^{-1}\underline{A} \geq 0.$$

Now assume that $v \in \mathbb{R}^n_+$, $w \in \mathbb{R}^m$ are such that $\tilde{A}v + Bw \geq 0$. By restricting attention to rows $\{i_1, \ldots, i_m\}$ (where $\underline{A} - \underline{B}F = 0$) we have that

$$(\underline{A} - \underline{B}F)v + \underline{B}w \geq 0 \Rightarrow \underline{B}w \geq 0 \Rightarrow w \geq 0,$$

as $\underline{B}^{-1} \geq 0$. We conclude that (A) holds.

To prove the converse we use the characterisation of (A) from Lemma 2.2. Suppose that (A) holds so that there exists $H, F \in \mathbb{R}^{m\times n}$ such that (a)–(d) hold. We also need a corollary to [16, Lemma 4.3, p. 68], which we repeat here. A real nonnegative $n \times m$ matrix $X$ of rank $m$ has a nonnegative left inverse if, and only if, $X$ contains an $m \times m$ monomial submatrix.

Assumptions (a) and (d) imply that $B$ has a nonnegative left inverse $H$ and thus $B$ must contain at least one (although possibly many) $m \times m$ positive monomial submatrix (submatrices), as must $H$. From the above arguments there must be a set of $m$ rows

$$\{i_1, \ldots, i_m\} \subseteq \{1, 2, \ldots, n\},$$

of $B$ that give rise to a monomial submatrix $\underline{B}$ where the corresponding columns $\{i_1, \ldots, i_m\}$ of $H$ must each have precisely one nonzero entry. The columns must each have at least one nonzero entry so that the product $HB$ does not have a zero column. They cannot have more than one else $HB = I_m$ cannot hold.

The equalities $HA = F$ and $HB = I_m$ together imply that $H(A - BF) = 0$ and as $H \geq 0$, $H \neq 0$, $A - BF \geq 0$ it follows that the rows $\{i_1, \ldots, i_m\}$ of $A - BF$ must be zero. Therefore, restricting to the rows $\{i_1, \ldots, i_m\}$ we have that

$$\underline{A} - \underline{B}F = 0, \tag{2.7}$$

where $\underline{A}$ is an $m \times n$ submatrix formed from rows $\{i_1, \ldots, i_m\}$ of $A$. From (2.7) it follows that $F = \underline{B}^{-1}\underline{A}$ which by construction yields $A - B\underline{B}^{-1}\underline{A} = A - BF \geq 0$, as required. □

`Matlab` code for verifying whether (A) holds for a given $(A, B)$ is available as online supplementary material. We comment here that (A) holds for any $A \geq 0$ in the single input case $B = b = c_i e_i$, $c_i > 0$ and the corresponding multiple input version case when $B$ is a combination of $e_i$, that is, $B = [c_{i_1}e_{i_1}, \ldots, c_{i_m}e_{i_m}]$ for positive $c_{i_k}$. These two cases are arguably the most important for applications. The following corollary interprets Lemma 2.1 in the single input case.

**Corollary 2.3.** *Let* $A \geq 0$ *with* $i$*th row denoted by* $r_i$ *and* $B = b$ *be given by*

$$b = \sum_{k=1}^{n} c_{i_k} e_{i_k} \quad \text{with } c_{i_k} > 0.$$

*Assumption* (A) *holds for* $(A, b)$ *if, and only if, there exists* $i_k \in \{i_1, \ldots, i_n\}$ *such that*

$$r_{i_j} - \frac{c_{i_j} r_{i_k}}{c_{i_k}} \geq 0, \quad \forall i_j \in \{i_1, \ldots, i_n\}, \tag{2.8}$$

*and in this case* $F = f^T = \frac{r_{i_k}}{c_{i_k}}$*, where* $i_k$ *is as in* (2.8).