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Robustness of stability through necessary and sufficient Lyapunov-like conditions for systems with a continuum of equilibria[☆]

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ABSTRACT

The equivalence between robustness to perturbations and the existence of a continuous Lyapunov-like mapping is established in a setting of multivalued discrete-time dynamics for a property sometimes called semistability. This property involves a set consisting of Lyapunov stable equilibria and surrounded by points from which every solution converges to one of these equilibria. As a consequence of the main results, this property turns out to always be robust for continuous nonlinear dynamics and a compact set of equilibria. Preliminary results on reachable sets, limits of solutions, and set-valued Lyapunov mappings are included.

1. Introduction

For a discrete-time dynamical system, this paper deals with a set which consists of Lyapunov stable equilibria, each of which is surrounded by points from where trajectories converge to one of these equilibria. Such equilibria were called *semistable* in [1]. Semistability in this sense was then studied in [2–5], and more. The term "semistability", in a sense related to partial stability and different from [1], was discussed in Russian control literature; see the survey [6]. The term was also used by [7] and related works to represent a property of an equilibrium weaker than Lyapunov stability, in the setting of abstract and set-valued dynamical systems. Some results of [7] are related to preliminary results here; see Remark 2.11. This paper uses the term *pointwise asymptotic stability* to characterize a set consisting of semistable, in the sense of [1], equilibria. The goal is to establish robustness of pointwise asymptotic stability to perturbations in dynamics and characterize this robustness through regularity of appropriate Lyapunov-like mappings.

Sufficient conditions for pointwise asymptotic stability, for differential equations [1] and then for differential inclusions [3], were given in terms of classical Lyapunov functions and non-tangent, to the set of equilibria, behavior of trajectories. A converse Lyapunov result for differential equations was obtained in [2]; this converse result does not result in a sufficient Lyapunov condition. A different approach, inspired by [8] where the decrease of a set-valued mapping was proposed as a sufficient condition for consensus, was pursued by the author in [9]. Necessary and sufficient Lyapunov-like conditions for pointwise asymptotic stability and a related invariance principle were given there in terms of a set-valued Lyapunov-like mapping. Some results of [9] are recalled in Section 3. Robustness of pointwise asymptotic stability has received limited treatment. The converse result of [2] was used in [10] to state robustness to higher-order perturbations under homogeneity assumptions. This robustness result included assumptions on Lyapunov stability of the equilibria for the perturbed, not just nominal, dynamics. A related result was given in [11] for a switching system.

For the classical concept of asymptotic stability, the equivalence of asymptotic stability to the existence of Lyapunov functions, with further relation between robustness of the asymptotic stability or regularity of the dynamics to the continuity or smoothness of Lyapunov functions, is well-appreciated. In particular, the equivalence of robustness of asymptotic stability of an equilibrium and the existence of a smooth Lyapunov function in the setting of nonlinear and multivalued dynamics was exhibited first in [12], in the setting of differential inclusions. This was later carried over to asymptotic stability of sets [13], to difference inclusions [14], and hybrid dynamics [15].

The contribution of this paper is showing the equivalence of robustness of pointwise asymptotic stability to the existence of continuous set-valued Lyapunov functions, in the setting of multivalued, but continuous in an appropriate sense, discrete-time dynamics and for a compact set of equilibria. This is shown in Theorem 4.3. Because continuous set-valued Lyapunov functions

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exist for pointwise asymptotically stable sets when the dynamics are continuous, for such dynamics the pointwise asymptotic stability of compact sets is always robust to sufficiently small perturbations. This is stated in Corollary 4.5 and appears to be a new result even in the single-valued case, where the dynamics are given by a continuous function.

The relevance of pointwise asymptotic stability for the analysis of consensus algorithms has been discussed, for example, in [2,10]. The issue of robustness of consensus algorithms has seen treatment in the literature, with the focus on convergence and not Lyapunov stability of consensus/equilibrium states and most often with robustness to changes of communication topology in time or to delays. Examples in Section 2 illustrate how state-dependent changes in communication topology fit in the framework of this paper and hint at applications of the robustness results to consensus problems. Furthermore, Example 2.5 suggests potential application of the results to analysis of optimization algorithms.

The paper is organized as follows. Section 2 introduces pointwise asymptotic stability and other basic concepts, and collects preliminary results on the behavior of solutions to a difference inclusion in the presence of a pointwise asymptotically stable closed set. In particular, results on continuous or semicontinuous dependence of the reachable set and of the limits of solutions on initial points are given. Section 3 introduces set-valued Lyapunov functions and employs them in necessary and sufficient conditions for pointwise asymptotic stability. A key observation here is that continuous set-valued dynamics lead to a continuous set-valued Lyapunov function. Section 4 states and proves the main result, Theorem 4.3.

2. Setting and basic results

Throughout the paper, a difference inclusion

$$x^+ \in F(x), \tag{1}$$

is considered, where $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a set-valued mapping, i.e., for every $x \in \mathbb{R}^n$, $F(x)$ is a subset of $\mathbb{R}^n$. The function $\phi : \mathbb{N}_0 \to \mathbb{R}^n$, where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, is a *solution to* (1) from the initial point $\xi \in \mathbb{R}^n$ if $\phi(0) = \xi$ and, for every $i \in \mathbb{N}$, $\phi(i) \in F(\phi(i-1))$. The set of all solutions to (1) from ξ is denoted as $\mathcal{S}(\xi)$. Given a set $C \subset \mathbb{R}^n$, $\mathcal{S}(C)$ is the set of all solutions to (1) from points in C, $\mathcal{S}(C) = \bigcup_{\xi \in C} \mathcal{S}(\xi)$.

One motivation for considering set-valued dynamics, following Krasovskii [16] and Filippov [17], is the link between set-valued regularization of discontinuous dynamics and the effect on such dynamics of perturbations, as shown in [18,19] for differential equations and in [20] for hybrid systems, which encompass difference equations and inclusions used here. The example below illustrates this.

Example 2.1. Let $x_1, x_2, \dots, x_I \in \mathbb{R}^m$ represent states of I agents. Suppose that each agent changes its position following the rule: find the average state of all agents, including myself, whose states differ less than 1 from my state and move half-way towards this average. This and similar dynamics have seen treatment in the literature, with the origins going back to [21,22]; see the extensive discussion in [23]. In the simple case of two one-dimensional agents, dynamics are given by the function $f(x)$ equal to

$$\begin{cases} (x_1, x_2) & \text{if } |x_1 - x_2| \geq 1 \\ \left(\frac{3}{4}x_1 + \frac{1}{4}x_2, \frac{1}{4}x_1 + \frac{3}{4}x_2\right) & \text{if } |x_1 - x_2| < 1. \end{cases}$$

The set-valued regularization of this discontinuous f is given by the set-valued mapping F whose graph is the closure of the graph of f. Alternatively, F is the "smallest" outer semicontinuous, as defined below, mapping such that $f(x) \in F(x)$ for every $x \in \mathbb{R}^n$. Explicitly, $F(x)$ is

$$\begin{cases} (x_1, x_2) & \text{if } |x_1 - x_2| > 1 \\ \left\{x_1, \frac{3x_1}{4} + \frac{x_2}{4}\right\} \times \left\{x_2, \frac{x_1}{4} + \frac{3x_2}{4}\right\} & \text{if } |x_1 - x_2| = 1 \\ \left(\frac{3x_1}{4} + \frac{x_2}{4}, \frac{x_1}{4} + \frac{3x_2}{4}\right) & \text{if } |x_1 - x_2| < 1. \end{cases}$$

The Cartesian product representing $F(x)$ when $|x_1 - x_2| = 1$ contains four points and represents the fact that under small perturbations, or measurement error if this is cast as a feedback control problem, agent x_1 can either move or remain stationary, and so can x_2. △

Let $M : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a set-valued mapping. Let $x \in \mathbb{R}^n$. Then M has a *nonempty value* at x if $M(x) \neq \emptyset$. M is *outer semicontinuous (osc) at* x if for every sequence $x_i \to x$, every convergent sequence $y_i \in F(x_i)$, one has $\lim_{i\to\infty} y_i \in F(x)$. It is *continuous at* x if, additionally, for every $y \in F(x)$, every sequence $x_i \to x$, there exist $y_i \in F(x_i)$ such that the sequence y_i converges and $\lim_{i\to\infty} y_i = y$. The mapping M is *locally bounded* at x if there exists a neighborhood U of x such that $F(U) = \bigcup_{z\in U} F(z)$ is bounded. If M has compact values and is *locally bounded at* x, then osc at x is equivalently described by: for every $\varepsilon > 0$ there exists $\delta > 0$ such that $F(x + \delta\mathbb{B}) \subset F(x) + \varepsilon\mathbb{B}$, which means that for every $z \in x + \delta\mathbb{B}$, $F(z) \subset F(x) + \varepsilon\mathbb{B}$. Here $\mathbb{B}$ is the closed unit ball in $\mathbb{R}^n$, and so $z \in x + \delta\mathbb{B}$ means that z is in a closed ball of radius δ around x, i.e., $\|z - x\| \leq \delta$. The additional condition for continuity of such M at x is: for every $\varepsilon > 0$ there exists $\delta > 0$ such that, for every $z \in x + \delta\mathbb{B}$, $F(x) \subset F(z) + \varepsilon\mathbb{B}$.

Throughout the paper, the following assumption is posed. It ensures, among other things, that solutions to (1) exist.

Assumption 2.2. The set-valued mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ has nonempty values and is locally bounded at every $x \in \mathbb{R}^n$.

For $J \in \mathbb{N}_0$, consider the finite-horizon reachable set

$$\mathcal{R}_{\leq J}(\xi) := \{\phi(j) \mid \phi \in \mathcal{S}(\xi), j \in \{0, 1, \dots, J\}\}.$$

When F is locally bounded, then $\mathcal{R}_{\leq J}$ is locally bounded, and then, if F is osc or continuous then so is $\mathcal{R}_{\leq J}$. This can be verified directly, but also follows from the representation $\mathcal{R}_{\leq J}(\xi) = \{\xi\} \cup F(\xi) \cup F^2(\xi) \cup \dots \cup F^J(\xi)$ and results about unions and compositions of set-valued mappings, [24, 4.31, 5.52]. The infinite-horizon reachable set

$$\mathcal{R}(\xi) = \bigcup_{J\in\mathbb{N}_0} \mathcal{R}_{\leq J}(\xi)$$

does not inherit regularity properties from F, in fact, $\mathcal{R}(\xi)$ can fail to have closed values even if F is a continuous function. Better regularity properties hold for the closure of the reachable set, i.e., the mapping $\overline{\mathcal{R}} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ given by

$$\overline{\mathcal{R}}(\xi) = \overline{\mathcal{R}(\xi)}.$$

2.1. Pointwise asymptotic stability

A set consisting of equilibria which are semistable in the terminology of [1], i.e., a set consisting of Lyapunov stable equilibria and surrounded by points from which solutions converge to some equilibrium in the set, will be called pointwise asymptotically stable. Below, rge ϕ denotes the range of the solution ϕ, so, for example, rge $\phi \subset a + \varepsilon\mathbb{B}$ means that $\phi(j) \in a + \varepsilon\mathbb{B}$ for every $j \in \mathbb{N}_0$.

Definition 2.3. The set $A \subset \mathbb{R}^n$ is *locally pointwise asymptotically stable* (PAS) for (1) if

- every $a \in A$ is *Lyapunov stable* for (1), that is, for every $a \in A$, every $\varepsilon > 0$ there exists $\delta > 0$ such that rge $\phi \subset a + \varepsilon\mathbb{B}$ for every $\phi \in \mathcal{S}(a + \delta\mathbb{B})$, and