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Singular perturbation approximation by means of a H^2 Lyapunov function for linear hyperbolic systems

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ABSTRACT

A linear hyperbolic system of two conservation laws with two time scales is considered in this paper. The fast time scale is modeled by a small perturbation parameter. By formally setting the perturbation parameter to zero, the full system is decomposed into two subsystems, the reduced subsystem (representing the slow dynamics) and the boundary-layer subsystem (standing for the fast dynamics). The solution of the full system can be approximated by the solution of the reduced subsystem. This result is obtained by using a H^2 Lyapunov function. The estimate of the errors is the order of the perturbation parameter for all initial conditions belonging to H^2 and satisfying suitable compatibility conditions. Moreover, for a particular subset of initial conditions, more precise estimates are obtained. The main result is illustrated by means of numerical simulations.

1. Introduction

Singular perturbations were introduced in control engineering in the late 1960s and have become a tool for analysis and design of control systems [1–4]. Singularly perturbed systems often occur naturally due to the presence of small parasitic parameters, for example, inductance in DC motors model, high gain amplifier in voltage regulators [5]. The singular perturbation method is a way of neglecting the fast transitions and considering them in a separate fast time scale. The significant advantage of this technique is to reduce the system order. Singularly perturbed partial differential equations have been considered in research works from late 1980s. Such systems are interesting since they describe many phenomena in various fields in physics and engineering, see [6] as a survey.

Tikhonov theorem is a fundamental tool for the analysis of singularly perturbed systems. It describes the limiting behavior of solutions of the perturbed system. Tikhonov theorem has been studied for finite dimensional systems modeled by ODEs in many research works [7,8]. The approximation of the full system by the reduced subsystem on a finite time interval requires only the exponential stability of the boundary-layer subsystem. Furthermore, the approximation on an infinite time interval is achieved based on the exponential stability of both subsystems.

In the previous work [9] it has been considered a Tikhonov theorem for linear singularly perturbed hyperbolic systems. The approximation has been achieved by using a L^2 Lyapunov function. Different from the previous work, we consider a H^2 Lyapunov function for a more sophisticated system which consists of the error between the slow dynamics of the full system and the reduced subsystem, the fast dynamics of the full system and the dynamics of the reduced subsystem in this work. The contribution of the present work is a more precise approximation result obtained by Lyapunov methods. More specifically, two cases are considered. In the first case, for any initial conditions belonging to H^2 satisfying suitable compatibility conditions, the difference between the slow dynamics of the full system and the reduced subsystem in L^2-norm is estimated of the order of the small perturbation parameter ϵ. Furthermore, the estimate of the fast dynamics in H^2-norm is also of the order of ϵ. In the second case, where the equilibrium point is chosen as the initial condition of the fast dynamics, the two estimates are obtained of the order of ϵ^2.

The paper is organized as follows. In Section 2, the singularly perturbed linear hyperbolic system under consideration is introduced, and the reduced subsystem is computed. Section 3 contains the main result on singular perturbation approximation of solutions for the full system by that for the subsystem. A numerical simulation is provided in Section 4 to illustrate the main result. Finally, concluding remarks end the paper.

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Notation. For a partitioned symmetric matrix M in $\mathbb{R}^{n\times n}$, the symbol $*$ stands for symmetric block. $M > 0$, $M < 0$ mean that M is positive definite and negative definite respectively. M^{-1} and $M^\top$ represent the inverse and the transpose matrix of M. $|\ |$ denotes the usual Euclidean norm and $\|\ \|$ is the associated matrix norm. $\|\ \|_{L^2}$ denotes the associated norm in $L^2(0,1)$ space, defined by $\|f\|_{L^2} = \left(\int_0^1 |f|^2 dx\right)^{\frac{1}{2}}$ for all functions $f \in L^2(0,1)$. Similarly, the associated norm in $H^2(0,1)$ space is denoted by $\|\ \|_{H^2}$, defined for all functions $f \in H^2(0,1)$, by $\|f\|_{H^2} = \left(\int_0^1 |f|^2 + |f_x|^2 + |f_{xx}|^2 dx\right)^{\frac{1}{2}}$. Following [10], we introduce the notation, $\rho_1(M) = \inf\{\|\Delta M \Delta^{-1}\|, \Delta \in D_{n,+}\}$, where $D_{n,+}$ denotes the set of diagonal positive matrices in $\mathbb{R}^{n\times n}$.

2. Linear singularly perturbed hyperbolic system of two conservation laws

Let us consider a 2×2 linear singularly perturbed hyperbolic system as follows

$$\begin{aligned} &y_t(x,t) + y_x(x,t) = 0, \\ &\epsilon z_t(x,t) + z_x(x,t) = 0, \end{aligned} \tag{1}$$

where $x \in [0,1]$, $t \in [0,+\infty)$, $y : [0,1] \times [0,+\infty) \to \mathbb{R}$, $z : [0,1] \times [0,+\infty) \to \mathbb{R}$ and $0 < \epsilon < 1$ is a small perturbation parameter.

We consider the following boundary conditions for system (1)

$$\begin{pmatrix} y(0,t) \\ z(0,t) \end{pmatrix} = G \begin{pmatrix} y(1,t) \\ z(1,t) \end{pmatrix}, \quad t \in [0,+\infty), \tag{2}$$

where $G = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix}$ is a 2×2 constant matrix.

Given two functions $y^0 : [0,1] \to \mathbb{R}$ and $z^0 : [0,1] \to \mathbb{R}$, the initial conditions are

$$\begin{pmatrix} y(x,0) \\ z(x,0) \end{pmatrix} = \begin{pmatrix} y^0(x) \\ z^0(x) \end{pmatrix}, \quad x \in [0,1]. \tag{3}$$

Remark 1. According to Proposition 2.1 in [10], for every $(y^0\ z^0)^\top \in H^2(0,1)$ satisfying the following compatibility conditions

$$\begin{pmatrix} y^0(0) \\ z^0(0) \end{pmatrix} = G \begin{pmatrix} y^0(1) \\ z^0(1) \end{pmatrix}, \tag{4}$$

$$\begin{pmatrix} y_x^0(0) \\ \frac{1}{\epsilon} z_x^0(0) \end{pmatrix} = G \begin{pmatrix} y_x^0(1) \\ \frac{1}{\epsilon} z_x^0(1) \end{pmatrix}, \tag{5}$$

the system (1) and (2) has a unique maximal classical solution $(y\ z)^\top \in C^0([0,+\infty), H^2(0,1))$.

Due to Section 2.1 in [11], for all $(y^0\ z^0)^\top \in L^2(0,1)$, there exists a unique maximal weak solution $(y\ z)^\top \in C^0([0,+\infty), L^2(0,1))$ to (1) and (2). ◇

The exponential stability of the linear system (1)–(2) in L^2-norm and H^2-norm is defined as follows

Definition 1. The linear system (1)–(2) is *exponentially stable to the origin in L^2-norm* if there exist $\gamma_1 > 0$ and $C_1 > 0$, such that for every $(y^0\ z^0)^\top \in L^2(0,1)$, the solution to the system (1)–(2) satisfies

$$\left\| \begin{pmatrix} y(.,t) \\ z(.,t) \end{pmatrix} \right\|_{L^2} \leqslant C_1 e^{-\gamma_1 t} \left\| \begin{pmatrix} y^0 \\ z^0 \end{pmatrix} \right\|_{L^2}, \quad t \in [0,+\infty).$$

The linear system (1)–(2) is *exponentially stable to the origin in H^2-norm* if there exist $\gamma_2 > 0$ and $C_2 > 0$, such that for every $(y^0\ z^0)^\top \in H^2(0,1)$ satisfying the compatibility conditions (4)–(5), the solution to the system (1)–(2) satisfies

$$\left\| \begin{pmatrix} y(.,t) \\ z(.,t) \end{pmatrix} \right\|_{H^2} \leqslant C_2 e^{-\gamma_2 t} \left\| \begin{pmatrix} y^0 \\ z^0 \end{pmatrix} \right\|_{H^2}, \quad t \in [0,+\infty).$$

Generalizing the approach in [5] to infinite dimensional systems, let us compute the reduced subsystem of the infinite dimensional systems. By setting $\epsilon = 0$ in system (1), we get

$$y_t(x,t) + y_x(x,t) = 0, \tag{6a}$$

$$z_x(x,t) = 0. \tag{6b}$$

Substituting (6b) into the boundary conditions (2) and assuming $G_{22} \neq 1$ yields

$$\begin{aligned} y(0,t) &= \left(G_{11} + \frac{G_{12}G_{21}}{1-G_{22}}\right) y(1,t), \\ z(.,t) &= \frac{G_{21}}{1-G_{22}} y(1,t). \end{aligned} \tag{7}$$

The reduced subsystem is then calculated as

$$\bar{y}_t(x,t) + \bar{y}_x(x,t) = 0, \quad x \in [0,1],\ t \in [0,+\infty), \tag{8}$$

with the boundary condition

$$\bar{y}(0,t) = G_r \bar{y}(1,t), \quad t \in [0,+\infty), \tag{9}$$

where $G_r = G_{11} + \frac{G_{12}G_{21}}{1-G_{22}}$. The bar is used to indicate that the variable belongs to the system with $\epsilon = 0$.

The initial condition of the reduced subsystem (8)–(9) is chosen as the same as for the full system (1)–(2), i.e.

$$\bar{y}^0(x) = y^0(x), \tag{10}$$

and the compatibility conditions for existence of H^2 solutions are given by

$$\begin{aligned} \bar{y}^0(0) &= G_r \bar{y}^0(1), \\ \bar{y}_x^0(0) &= G_r \bar{y}_x^0(1). \end{aligned} \tag{11}$$

Let us recall the stability result for linear hyperbolic system (1)–(2) given in [10,12].

Theorem 1 ([10,12]). *If $\rho_1(G) < 1$ (resp. $\rho_1(G_r) < 1$), the linear system (1)–(2) (resp. the reduced subsystem (8)–(9)) is exponentially stable to the origin in L^2-norm and H^2-norm.*

3. Tikhonov approach for linear singularly perturbed hyperbolic systems

This section presents an approximation theorem for system (1)–(2). More precisely, the difference of the solution between the full system (1)–(2) and the reduced subsystem (8)–(9) is firstly estimated of the order of ϵ. Then for particular initial conditions, it is estimated of the order of ϵ^2. Therefore, the solution of the full system can be approximated by the solution of the reduced subsystem. A H^2 Lyapunov function is used to prove this result.

Let us first perform a change of variable

$$\eta = y - \bar{y},$$

where y is the solution of the full system and $\bar{y}$ is the solution of the reduced subsystem.

By considering the fast dynamics z in (1)–(2) and the dynamics $\bar{y}$ in (8)–(9), let us study the following system

$$\begin{aligned} &\eta_t + \eta_x = 0, \\ &\epsilon z_t + z_x = 0, \\ &\bar{y}_t + \bar{y}_x = 0, \end{aligned} \tag{12}$$