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On the tightness of linear policies for stabilization of linear systems over Gaussian networks

Ali A. Zaidi [a,*], Serdar Yüksel [b], Tobias J. Oechtering [c], Mikael Skoglund [c]

[a] Ericsson Research, Stockholm, Sweden

[b] Department of Mathematics and Statistics, Queen's University, Kingston, Ontario, Canada

[c] School of Electrical Engineering and the ACCESS Linnaeus Center, KTH Royal Institute of Technology, Stockholm, Sweden

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ABSTRACT

In this paper, we consider stabilization of multi-dimensional linear systems driven by Gaussian noise controlled over parallel Gaussian channels. For such systems, it has been recognized that for stabilization in the sense of asymptotic stationarity or stability in probability, Shannon capacity of a channel is an appropriate measure on characterizing whether a system can be made stable when controlled over the channel. However, this is in general not the case for quadratic stabilization. On a related problem of joint-source channel coding, in the information theory literature, the source-channel matching principle has been shown to lead to optimality of uncoded or analog transmission and when such matching conditions occur, it has been shown that capacity is also a relevant figure of merit for quadratic stabilization. A special case of this result is applicable to a scalar LQG system controlled over a scalar Gaussian channel. In this paper, we show that even in the absence of source-channel matching, to achieve quadratic stability, it may suffice that information capacity (in Shannon's sense) is greater than the sum of the logarithm of unstable eigenvalue magnitudes. In particular, we show that periodic linear time varying coding policies are optimal in the sense of obtaining a finite second moment for the state of the system with minimum transmit power requirements for a large class of vector Gaussian channels. Our findings also extend the literature which has considered noise-free systems.

1. Problem formulation

Consider the following linear time invariant system:

$$\bar{X}_{t+1} = A\bar{X}_t + B\bar{U}_t + \bar{W}_t, \quad t \in \mathbb{N}, \tag{1}$$

where $\bar{X}_t \in \mathbb{R}^n$ is a state process, $\bar{U}_t \in \mathbb{R}^n$ is a control process, $\bar{W}_t \in \mathbb{R}^n$ is an independent and identically distributed sequence of Gaussian random variables with zero mean and covariance K_W. The system matrix A and the input matrix B are of appropriate dimensions and we assume that the pair (A, B) is controllable. Let $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$ be the eigenvalues of the system matrix A. Without loss of generality we assume that all the eigenvalues of A are outside the unit disc ($1 \le |\lambda_i| < \infty$ for all i), i.e., all modes are unstable. The initial state of the system X_0 is assumed to be a random variable with zero mean and covariance Λ_0 with Trace $\{\Lambda_0\} < \infty$. The initial state X_0 is assumed to be independent of the plant noise variable $\bar{W}_t$. Consider the scenario depicted in Fig. 1, where a sensor observes an n-dimensional state process and transmits it to a remote controller over m parallel independent Gaussian channels. At any time instant t, $S_t := [s_{1,t}, s_{2,t}, \dots, s_{m,t}]$ and $R_t := [r_{1,t}, r_{2,t}, \dots, r_{m,t}]$ are the input and output of the channel, where $r_{i,t} = s_{i,t} + z_{i,t}$ and $z_{i,t} \sim \mathcal{N}(0, N_i)$ are zero mean white Gaussian noise components with $N_1 \le N_2 \le \cdots \le N_m$. We assume that there is a noiseless causal feedback link from the controller to the sensor and the plant. Let $f_t : \mathbb{R}^{(n+m)t+n} \to \mathbb{R}^m$ denote the sensing policy such that $S_t = f_t(X_{[0,t]}, R_{[0,t-1]})$, where $X_{[0,t]} := \{X_0, X_1, \dots, X_t\}$ and the sensor is assumed to have an average transmit power constraint $\mathbb{E}[\|S_t\|^2] \le P_S$. Further, let $\pi_t : \mathbb{R}^{m(t+1)} \to \mathbb{R}^n$ be the controller policy, then we have $U_t = \pi_t(R_{[0,t]})$. The common goal of the sensor and the controller is to stabilize the system (1) in the mean square sense, defined as follows.

Definition 1.1 (*[1, Definition 2.2]*). A system is said to be *mean square stable* if there exists $M < \infty$ such that $\sup_t \mathbb{E}[\|X_t\|^2] < M$.

* Corresponding author.
E-mail address: zaidi@kth.se (A.A. Zaidi).

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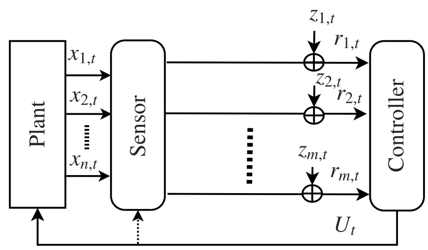

Fig. 1. Control over parallel Gaussian channels.

Literature review. Stabilization of linear systems over communication channels has been studied in [2–4,1,5–17]. If the goal is stabilization in the sense of asymptotic mean stationarity [18] or similar notions such as stability in probability [1,6], Shannon capacity is the right measure on what is possible or not (see [18] for a detailed account of such results in the literature). But if the goal is stabilization in the sense of having finite second moments, then Shannon capacity may not the right measure [1].

A related, but different problem, is the joint source-channel coding problem. In this context, the Gaussian source-channel setting is an important special case due to the possible mean-square optimality of linear coding policies as a consequence of what is known as source-channel matching [19–21]. Using the data-processing theorem of information theory and dynamic programming (see e.g. [20,22,18]), it can be shown that when source-channel matching occurs, linear coding policies for controlled linear Gaussian sources are optimal for the minimization of quadratic distortion measures across Gaussian channels. However, such a source-channel matching does not apply in a large class of settings involving multi-dimensional sources and channels [23–26]. Along a similar context, we note that recent work [27] has obtained structural properties of channels which can be used to realize optimal causal channel codes for a class of multi-dimensional Gaussian sources with memory.

In the control literature, these problems have been considered where the sufficiency of information capacity[1] being greater than a lower bound has been observed in a class of settings [10–12], which, however, consider *noise-free plants*. It is observed in [11] that LTI schemes are not optimal for stabilization over parallel channels. For optimal encoding in the unmatched case, linear encoding is also not optimal in general [29,30]. Even in the class of memoryless coding schemes, linear coding is not optimal for the transmission of memoryless Gaussian sources over memoryless Gaussian channels with quadratic distortion measures [23,25,31,32].

That a scalar Gaussian channel allows for stability when its information capacity is greater than the sum of the logarithm of unstable eigenvalue magnitudes of a linear system, not only in the sense of ergodicity but also in the sense of quadratic stability, is not surprising. The reason for this argument is that for such channels, the data processing inequality arguments lead to the optimality of linear coding and decoding policies for the minimization of the quadratic estimation error for the state (see *Chapter 11* in [18]). One can also show that for a scalar Gaussian channel, the error exponent with feedback is not bounded [33–35] and using arguments in [1,14,18], one expects that quadratic stability is possible even for systems driven by unbounded noise.

Results on controlling a vector linear system over a scalar Gaussian channel have been obtained in [36] confirming this line of thought, where linear time-varying policies have been shown to be sufficient for mean-square stability. However, there does not exist result in the literature that considers noisy multi-dimensional linear systems controlled over multi-dimensional Gaussian channels. For such channels, in general, the information theoretic approach based on the data-processing inequality does not lead to tight bounds on optimal joint-source-channel coding schemes, unlike the scalar case.

Contributions of the paper: In this paper, we consider quadratic (second moment) stabilization of multi-dimensional linear systems (sources) represented by (1) over vector-valued Gaussian channels. We show that for a large class of source-channel pairs, information capacity being greater than the sum of the logarithm of unstable eigenvalue magnitudes of the linear system (1) is sufficient for quadratic stability and linear sensing and control schemes are optimal, even when the source-channel matching principle does not hold.

In the literature, stabilization results have been presented for *noiseless* multi-dimensional plants over multidimensional channels in [11,12,36,37] and for *noisy* multi-dimensional plants over *scalar* channels in [36]. Our paper extends these results to more general setups and establishes optimality of linear sensing and control schemes for the moment stabilization of a wide class of *noisy* linear plants over vector Gaussian channels.[2]

2. Sufficient conditions and a linear time-varying scheme

We have the following sufficiency result.

Theorem 2.1. *The system (1) can be mean square stabilized over m parallel independent Gaussian channels using a linear scheme if there exist $f_{ij} \in \mathbb{Q}$ such that $f_{ij} \geq 0$, $\sum_{j=1}^{m} f_{ij} \leq 1$, $\sum_{i=1}^{n} f_{ij} \leq 1$ and*

$$\log\left(|\lambda_i|\right) < \sum_{j=1}^{m} f_{ij} C_j, \quad \forall i \in \{1, 2, \ldots, n\}, \tag{2}$$

where λ_i are eigenvalues of the system matrix A in (1) and $C_j := \frac{1}{2}\log(1 + \frac{P_j}{N_j})$ is the information capacity of jth channel.

Proof. For the proof, we propose a periodic linear time varying scheme sensing and control scheme. We first give the scheme for a system with invertible input matrix B, assuming that $B = I$ in (1): Consider that the control actions in (1) are taken periodically after every K time steps, i.e., at $t = lK - 1$ for $l \in \mathbb{N}$ ($U_t = 0$ for $t \neq lK - 1$). Under this control strategy, the state equation at $t = lK$ is given by

$$\bar{X}_{t+K} = A^K \bar{X}_t + \bar{U}_{t+K-1} + \sum_{i=0}^{K-1} A^{K-i-1} \bar{W}_{t+i}. \tag{3}$$

For $A^K \in \mathbb{R}^{n \times n}$ there exist a real non-singular matrix T and a real matrix $\tilde{A}$ such that $\tilde{A} = T^{-1} A^K T = \operatorname{diag}\left[J_1, \ldots, J_p\right]$, where J_p is a Jordan block of dimension (algebraic multiplicity) n_p [39]. A Jordan block $J_p \in \mathbb{R}^{n_p \times n_p}$ associated with a real eigenvalue λ of multiplicity n_p has the following form:

$$J_p = \begin{pmatrix} \lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{pmatrix}, \tag{4}$$

[1] The definition of information capacity for Gaussian channels can be found in page 263 in [28].

[2] Part of results without proofs have been included in a book chapter that provides an overview of some recent results on stabilization and control over Gaussian networks [38].