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On the equivalence between global recurrence and the existence of a smooth Lyapunov function for hybrid systems

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ABSTRACT

We study a weak stability property called recurrence for a class of hybrid systems. An open set is recurrent if there are no finite escape times and every complete trajectory eventually reaches the set. Under sufficient regularity properties for the hybrid system we establish that the existence of a smooth, radially unbounded Lyapunov function that decreases along solutions outside an open, bounded set is a necessary and sufficient condition for recurrence of that set. Recurrence of open, bounded sets is robust to sufficiently small state dependent perturbations and this robustness property is crucial for establishing the existence of a Lyapunov function that is smooth. We also highlight some connections between recurrence and other well studied properties like asymptotic stability and ultimate boundedness.

1. Introduction

Hybrid systems are a class of dynamical systems that combine continuous-time evolution and discrete-time events. Several frameworks have been proposed in the literature for analysis of hybrid systems. We refer the reader to [1–3] for details. Converse Lyapunov theorems are used to establish the equivalence between asymptotic stability properties and the existence of Lyapunov-like functions that satisfy certain decrease conditions along solutions. Applications of converse theorems in stabilization and robust stability analysis can be found in [4–6]. For continuous-time systems converse theorems related to asymptotic stability are established in [6–8]. See [9–11] for similar results in the discrete-time case. Converse theorems for asymptotic stability of compact sets for a class of hybrid systems has been established in [12,13]. In this paper, we establish a converse theorem for a similar class of hybrid systems considered in [12] but for a weaker property called recurrence.

Recurrence of an open, bounded set is a weak stability property that is frequently studied in the literature for stochastic systems. We refer the reader to [14–16] for details. Loosely speaking, recurrence of an open, bounded set implies that solutions visit the set infinitely often with probability one. It is a weaker notion of stability compared to probabilistic notions of asymptotic stability but nevertheless useful in many applications. In particular, for the problem of RF-source seeking using Micro Aerial Vehicles (MAV) considered in [17,18], it is shown that asymptotic properties like convergence are difficult to establish in the presence of persistent disturbances and recurrence proves to be a useful alternative.

In this paper we study the recurrence property not for stochastic systems, but for a class of non-stochastic hybrid systems. To the best of the authors' knowledge this property has not been studied in detail for non-stochastic hybrid systems. Although recurrence is a weak property in the context of stochastic systems, we show that recurrence of an open, bounded set is actually equivalent to establishing uniform ultimate boundedness of solutions for non-stochastic systems. A similar observation is made for discrete-time deterministic systems in [15], although it is noted that such an equivalence does not hold true for recurrence in stochastic systems. The importance of ultimate boundedness in control design for uncertain systems is explained in [19,20]. We also note that recurrence-like properties are studied for deterministic systems in [21, Chapter 7] and [22, Chapter 1].

Necessary and sufficient conditions for global recurrence in terms of Lyapunov functions are established in this work. For discrete-time stochastic systems with non-unique solutions, i.e., stochastic difference inclusions, the results in [23] establish the equivalence between recurrence of an open, bounded set and the existence of a smooth Lyapunov function that decreases on average outside the recurrent set. This is achieved by establishing robustness of recurrence to various state dependent perturbations. We follow a similar approach in this paper to establish a converse Lyapunov theorem, but for a class of non-stochastic

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hybrid systems. A converse theorem for a stronger version of recurrence called positive recurrence is established for discrete-time stochastic systems in [15] and in [14, Thm 3.26] for a class of switching diffusion processes.

Converse theorems for recurrence in discrete-time deterministic systems is developed in [15, Thm 11.2.1] although the Lyapunov function generated is merely upper semicontinuous. Using the robustness of recurrence to various state dependent perturbations we construct a smooth Lyapunov function for the converse theorem. Robustness of recurrence to sufficiently small perturbations is due to the hybrid system satisfying good regularity properties. Without the regularity properties, it is not guaranteed that recurrence or other stability properties are robust. We exploit the robustness to go from a preliminary non-smooth Lyapunov function to a smooth Lyapunov function for recurrence by utilizing the construction in [12]. We also make connections of this property to asymptotic stability and ultimate boundedness. In [24] a robust boundedness problem is studied for continuous-time systems that uses the notion of first hitting times to certain forward invariant compact sets. In this paper, we will use similar tools, but to study the recurrence property. Finally, recurrence for non-stochastic systems has extra consequences compared to the stochastic counterpart. This paper will also briefly highlight some of these differences.

The rest of the paper is organized as follows. Section 2 presents the basic notation and definitions to be used in the paper. Section 3 introduces the hybrid systems framework that will be considered in the rest of the paper. Section 4 presents the definition of recurrence and its uniform version. Section 5 makes connections between recurrence and other well studied properties. Section 6 presents the main results. Section 7 presents an equivalent characterization of recurrence which will be used to prove the main results of the paper. The proof of the converse theorem is presented in Section 7. Section 9 presents some concluding comments and future work.

2. Notation and basic definition

For a closed set $S \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$, $|x|_S := \inf_{y\in S} |x - y|$ is the Euclidean distance of x to S. Let $\mathbb{B}$, $\mathbb{B}^o$ denote the closed and open unit ball in $\mathbb{R}^n$. Given a closed set $S \subset \mathbb{R}^n$ and $\epsilon > 0$, $S + \epsilon\mathbb{B}$ represents the set $\{x \in \mathbb{R}^n : |x|_S \le \epsilon\}$. $\mathbb{R}_{\ge 0}$ denotes the non-negative real numbers; $\mathbb{Z}_{\ge 0}$ denotes the non-negative integers. For $c \ge 0$ and a function $V : \mathbb{R}^n \to \mathbb{R}_{\ge 0}$, $L_V(c) := \{x \in \mathbb{R}^n : V(x) = c\}$. A set-valued mapping $M : \mathbb{R}^p \rightrightarrows \mathbb{R}^n$ is *outer semicontinuous* if, for each $(x_i, y_i) \to (x, y) \in \mathbb{R}^p \times \mathbb{R}^n$ satisfying $y_i \in M(x_i)$ for all $i \in \mathbb{Z}_{\ge 0}$, $y \in M(x)$. A mapping M is *locally bounded* if, for each bounded set $K \subset \mathbb{R}^p$, $M(K) := \bigcup_{x\in K} M(x)$ is bounded. A function $\Psi : \mathbb{R}^n \to \mathbb{R}_{\ge 0}$ is *upper semicontinuous* if for every sequence $\{x_i\}_{i=0}^{\infty}$ such that $x_i \to x$, we have $\limsup_{i\to\infty} \Psi(x_i) \le \Psi(x)$. For $S \subset \mathbb{R}^n$, the symbol $\mathbb{I}_S$ denotes the indicator function of S i.e., $\mathbb{I}_S(x) = 1$ for $x \in S$ and $\mathbb{I}_S(x) = 0$ otherwise. For vectors $f_1, f_2 \in \mathbb{R}^n$, $\langle f_1, f_2\rangle := f_1^T f_2$ denotes the inner product.

3. Preliminaries on hybrid systems

We follow the mathematical framework in [1] for modeling hybrid systems. As explained in [25, Chapter 1] other models for describing hybrid systems can be encompassed within the framework of [1]. So we consider a class of stochastic hybrid systems considered in [1] with a state $x \in \mathbb{R}^n$ written formally as

$$\dot{x} \in F(x), \quad x \in C \tag{1a}$$

$$x^+ \in G(x), \quad x \in D, \tag{1b}$$

where $C, D \subset \mathbb{R}^n$ represent the flow and jump sets (where continuous and discrete evolution of the state is permitted) respectively and F, G represent the flow and jump maps respectively. In essence, the continuous-time dynamics is modeled by a differential inclusion and the discrete-time dynamics is modeled by a difference inclusion. We consider a very general class of hybrid systems modeled by set-valued mappings as opposed to single-valued mappings as set-valued mapping arise naturally in the context of robustness analysis and study of ISS properties of hybrid systems with inputs. We refer the reader to [12,25,1] for more details.

The solution concept for systems of the form (1) is explained in detail in [25, Chapter 2]. We define solutions on a generalized time domain that uses two variables t, j to keep track of the continuous evolution of the state and the number of jumps elapsed. To define solutions to (1) we require the notion of a *hybrid time domain*: a subset E of $(\mathbb{R}_{\ge 0} \times \mathbb{Z}_{\ge 0})$, which is the union of infinitely many intervals of the form $[t_j, t_{j+1}] \times \{j\}$, where $0 = t_0 \le t_1 \le t_2 \le \cdots$, or finitely many of such intervals, with the last one possibly of the form $[t_j, t_{j+1}] \times \{j\}$, $[t_j, t_{j+1}) \times \{j\}$, or $[t_j, \infty) \times \{j\}$. A function $\phi : E \to \mathbb{R}^n$ that maps a hybrid time domain to the Euclidean space and for which $t \mapsto \phi(t, j)$ is locally absolutely continuous for fixed j is called a *hybrid arc*.

A hybrid arc is a *solution* to (1) if $\phi(0, 0) \in C \cup D$ and:

(1) for all $j \in \mathbb{Z}_{\ge 0}$ and almost all t such that $(t, j) \in \operatorname{dom}\phi$: $\phi(t, j) \in C$, $\dot{\phi}(t, j) \in F(\phi(t, j))$
(2) for all $(t, j) \in \operatorname{dom}\phi$ such that $(t, j + 1) \in \operatorname{dom}\phi$: $\phi(t, j) \in D$, $\phi(t, j + 1) \in G(\phi(t, j))$.

A solution to the hybrid system is called *maximal* if it cannot be extended, and *complete* if its domain is unbounded. We will represent the hybrid system through its data as

$$\mathcal{H} := (C, F, D, G). \tag{2}$$

We denote by $\mathcal{S}_{\mathcal{H}}(K)$ the set of all maximal solutions starting from the set $K \subset \mathbb{R}^n$ for the hybrid system $\mathcal{H}$. We assume throughout the paper that $\mathcal{H}$ satisfies certain regularity properties listed below.

Standing Assumption 1. The data $\mathcal{H}$ of the hybrid system satisfies the following conditions:

(1) The sets $C, D \subset \mathbb{R}^n$ are closed.
(2) The mapping F is outer semicontinuous, locally bounded, convex valued and non-empty on C.
(3) The mapping G is outer semicontinuous, locally bounded and non-empty on D.

If F, G are single-valued mappings, then Assumption 1 reduces to the mappings f, g being continuous on C and D respectively. The main motivation for such assumptions is to ensure that stability properties are robust for the hybrid system. The robustness of asymptotic stability to sufficiently small state dependent perturbations under the conditions of Assumption 1 is established in [12]. The system (1a) is said to have no finite escape times if there are no solutions of (1a) that escape to infinity at a finite time. In the rest of this paper, we will establish similar equivalences for the weaker property of recurrence and also illustrate using examples cases where such equivalences can fail due to the conditions of Assumption 1 not being satisfied.

4. Recurrence and uniform recurrence

In this section we define the notion of recurrence for sets.

Definition 1. A set $\mathcal{O} \subset \mathbb{R}^n$ is said to be globally recurrent for the hybrid system $\mathcal{H}$ in (2) if there are no finite escape times for (1a) and for each complete solution $\phi \in \mathcal{S}_{\mathcal{H}}(C \cup D)$, there exists $(t, j) \in \operatorname{dom}\phi$ such that $\phi(t, j) \in \mathcal{O}$.