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Stability of an analog optimization circuit for quadratic programming

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ABSTRACT

We study the stability of an analog optimization circuit that solves quadratic programming (QP) problems. The circuit dynamics are modeled as a switched affine system. A piece-wise quadratic Lyapunov function and the KYP lemma are used to derive the stability criterion. The stability criterion characterizes the range of critical circuit parameters for which the QP circuit is globally exponentially stable.

1. Introduction

Our renewed interest in analog optimization stems from the need to achieve a low latency solution for Model Predictive Control (MPC) [1]. In MPC at each sampling time, starting at the current state, an open-loop optimal control problem is solved over a finite horizon. The optimal command signal is applied to the process only during the following sampling interval. At the next time step a new optimal control problem based on new measurements of the state is solved over a shifted horizon. The optimal solution relies on a dynamic model of the process, respects input and output constraints, and minimizes a performance index. When the model is linear and the performance index is based on the two-norm, the resulting optimization problem can be cast as a quadratic program (QP), where the state enters the right hand side (rhs) of the constraints.

In [2] we presented the design of an analog QP circuit. We showed that its equilibrium voltages are the QP optimizers. The proposed circuit achieves lower latency and is simpler than the designs in the early analog optimization work in [3–5].

In this paper we study the dynamics of the circuit. Circuits that combine linear dynamics and switching elements have been extensively studied in the past [6,7]. We describe the circuit as continuous-time piecewise affine system with restricted switching logic and we derive a criterion for the exponential stability using a piecewise quadratic Lyapunov function. Stability of a piecewise affine system can be shown by numerically solving an appropriate LMI [8–10,6]. In this paper we exploit the structure of the circuit to show that the Lyapunov function exists for a range of critical circuit parameters. We make use of an eigenvalue decomposition and KYP lemma to derive the circuit parameter bound. Our results allow to quantify the maximum circuit speed as a function of the circuit parasitic effects.

Although we address a particular system in this paper, the methodology is of broader interest. This is because the resistor network in our system exhibits the general structure of diffusive coupling — an area of extensive research activity in dynamical systems. In addition, our system has particular features, such as state dependent switching of the coupling and nonuniform bias terms affecting the subsystems. Our methodology accounts for these features and may prompt further research for broader diffusively coupled systems with similar characteristics.

The paper is organized as follows. For the sake of better readability, a description of how to construct an analog circuit from a given QP is presented in Section 2. For more details and experimental results we refer the reader to the description in [2]. Section 3 presents stability analysis of the circuit and the main result of the paper. Concluding remarks are presented in Section 4.

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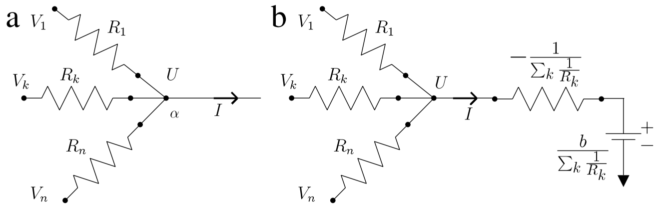

Fig. 1. (a) A node with k connected wires. (b) Equality enforcing circuit consisting of n resistors R_k, a negative resistance and a reference voltage.

2. QP analog circuit

Consider the quadratic programming (QP) problem

$$\min_{V=[V_1,\ldots,V_n]^T} \quad V^TQV \tag{1a}$$

$$\text{s.t.} \quad A_{\text{eq}}V = b_{\text{eq}} \tag{1b}$$

$$A_{\text{ineq}}V \le b_{\text{ineq}} \tag{1c}$$

where $V_1, \ldots, V_n$ are the optimization variables, b_{eq} and b_{ineq} are column vectors, $Q \succ 0$, and A_{ineq} and A_{eq} are matrices.

Without loss of generality, we assume that A_{ineq} and A_{eq} have non-negative entries. Indeed, a QP (1) with negative entries can be transformed into this form by introducing an auxiliary vector $\bar{V}$ as follows:

$$\begin{aligned} &\min_{\bar{V},V} \quad V^TQV \\ &\text{s.t.} \quad A_{\text{eq}}^+V + A_{\text{eq}}^-\bar{V} = b_{\text{eq}},\; A_{\text{ineq}}^+V + A_{\text{ineq}}^-\bar{V} \le b_{\text{ineq}} \\ &\qquad V + \bar{V} = 0, \end{aligned}$$

where A_{ineq} and A_{eq} are split into positive and negative parts ($A_{\text{ineq}} = A_{\text{ineq}}^+ - A_{\text{ineq}}^-$ and $A_{\text{eq}} = A_{\text{eq}}^+ - A_{\text{eq}}^-$).

In the beginning of this section we present the basic building blocks which will be later used to create a circuit that solves problem (1). The first basic block enforces equality constraints of the form (1b). The second building block enforces inequality constraints of the form (1c). The last basic block implements the cost function.

2.1. Equality constraint

Consider the circuit depicted in Fig. 1(a). In this circuit n wires are connected to a common node. We call this common node α, its potential U and the current that exits this node I. Kirchhoff's current law (KCL) implies

$$\sum_{k=1}^{n} I_k = \sum_{k=1}^{n} \frac{V_k - U}{R_k} = I, \tag{2}$$

where V_k is the potential of node k, R_k is the resistance between node k and the node α. Eq. (2) can be written as an equality constraint on potentials V_k:

$$\sum_{k=1}^{n} \frac{V_k}{R_k} = I + U\sum_{k=1}^{n} \frac{1}{R_k}. \tag{3}$$

If we can set the right hand side (rhs) of (3) to any desired value b, then (3) enforces an equality constraint on a linear combination of V_k variables. Therefore, every equality constraint (1b) can be implemented by assigning

$$U = \frac{b - I}{\sum\limits_{k=1}^{n} \frac{1}{R_k}}. \tag{4}$$

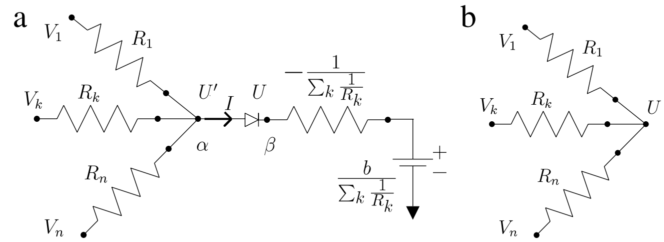

Fig. 2. (a) Inequality enforcing circuit. (b) Quadratic cost circuit.

Eq. (4) together with (3) yields

$$\begin{bmatrix} \frac{1}{R_1} & \cdots & \frac{1}{R_n} \end{bmatrix} \begin{bmatrix} V_1 \\ \vdots \\ V_n \end{bmatrix} = b \tag{5}$$

and the circuit implementing (5) is shown in Fig. 1(b).

2.2. Inequality constraint

Consider the circuit shown in Fig. 2(a). Similarly to the equality constraint circuit, n wires are connected to a common node α. Its potential is U' and the current exiting this node is I. Kirchhoff's current law (KCL) implies (2).

An ideal diode connects node α to node β. The potential of node β is U. The diode enforces $U' \le U$. In Fig. 2(a), the voltage U can be computed as follows

$$U = \frac{b - I}{\sum\limits_{k=1}^{n} \frac{1}{R_k}} \ge U'. \tag{6}$$

Eq. (2) and $U' \le U$ yield

$$\sum_{k=1}^{n} \frac{V_k}{R_k} = I + U'\sum_{k=1}^{n} \frac{1}{R_k} \le I + U\sum_{k=1}^{n} \frac{1}{R_k} = b, \tag{7}$$

which can be compactly rewritten as

$$\begin{bmatrix} \frac{1}{R_1} & \cdots & \frac{1}{R_n} \end{bmatrix} \begin{bmatrix} V_1 \\ \vdots \\ V_n \end{bmatrix} \le b. \tag{8}$$

2.3. Quadratic cost function

Let $A = \begin{bmatrix} A_{\text{eq}} \\ A_{\text{ineq}} \end{bmatrix}$ be the matrix of constraint coefficients. By composing the elementary circuits of the previous section we can design an analog circuit which implements the constraints $A_{\text{eq}}V = b_{\text{eq}}$ and $A_{\text{ineq}}V \le b_{\text{ineq}}$. By using simple energy arguments, in [2] it was shown that such circuit would minimize a cost function V^TQ_AV where

$$Q_A = \text{diag}(\mathbf{1}^TA) - A^T\,\text{diag}\left(\mathbf{1}^TA^T\right)^{-1}A. \tag{9}$$

In general, this cost function is different from the desired cost Q. However, it is possible to add redundant constraints of the form $A_{\text{augm}}V < \infty$, which are always inactive and have no effect on a feasible set of the problem (1). By doing so the cost matrix can be shaped in a way that $Q_{A'} = kQ$, where $A' = \begin{bmatrix} A \\ A_{\text{augm}} \end{bmatrix}$, $k > 0$ is a scalar, and $Q_{A'} = \text{diag}(\mathbf{1}^TA') - A'^T\,\text{diag}\left(\mathbf{1}^TA'^T\right)^{-1}A'$ (see [2] for additional details). The redundant constraints are implemented using a simple circuit depicted in Fig. 2(b), i.e, a special case of the inequality circuit, without the diode and the negative resistor.