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Computation of feedback control for time optimal state transfer using Groebner basis☆

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ABSTRACT

Computation of time optimal feedback control law for a controllable linear time invariant system with bounded inputs is considered. Unlike a recent paper by the authors, the target final state is not limited to the origin of state-space, but is allowed to be in the set of constrained controllable states. Switching surfaces are formulated as semi-algebraic sets using Groebner basis based elimination theory. Using these semi-algebraic sets, a nested switching logic is synthesized to generate the time optimal feedback control. However, the optimal control law enforces an unavoidable limit cycle in the time-optimal trajectory for most non-origin target points. The time-period of this limit-cycle is dependent on the target position. This dependence is algebraically characterized and a method to compute the time-period of the limit-cycle is provided. As a natural extension, the set of constrained controllable states is also computed.

1. Introduction

Recent advances in computer algebra packages have facilitated application of tools from algebraic geometry (in particular Groebner basis) to solve important problems in control theory (see [1–3]). In this work, we apply Groebner basis methods to the problem of computation of feedback control (from a constrained set $|u| \leq 1$) for the minimum time state-transfer of a linear time invariant (LTI) system. Pontryagin's minimum principle (PMP) provides the open loop optimal solution, according to which, the input switches between extreme admissible values ± 1 [4]. The feedback solution, however, requires knowledge of the so called *switching surfaces* in the state-space. If the equations for the switching surfaces (in terms of the state variables) can be computed, then it is possible to synthesize the optimal feedback using the sign of the current state with respect to switching surface. A partial solution to this synthesis problem was provided in [5] for a class of systems with positive eigenvalues and later on extended to non-zero distinct rational eigenvalues in [6]. In these papers, a time optimal feedback control law was synthesized (using Groebner basis) which transfers admissible initial states to the *origin* and keeps the state-trajectory arbitrarily close to the origin. In this article, we consider a class of systems with distinct and rational eigenvalues and extend our earlier results from [5,6] for minimum time state transfer to *non-origin* target points.

Optimal feedback is preferred over open loop solutions for robustness to disturbances as well as reduction in real time computational load. Time optimal feedback control has been employed, among others, in space-craft attitude control [7], robotic manipulators [8], pursuit evasion games [9] and multi-agent consensus tracking [10]. Control to non-origin targets is desirable when linearized equations of the corresponding non-linear system are valid over a wide range of operating conditions and tracking of different set points (not necessarily the equilibrium point) may be needed. Such situations occur regularly in a variety of applications (e.g. see [11] and references therein). The analytical foundations for the switching surfaces in time optimal control were laid out in [12,13] and later extended for non-origin target points in [14–16].

In this article, we use algebraic geometry techniques (i.e., Groebner basis based implicitization algorithm) for computing the semi-algebraic representation of switching surfaces (dependent purely on state-variables). We also construct a nested switching logic using the computed expressions for these surfaces. Interestingly, unlike the earlier case (i.e., [5,6]), the state-trajectory is forced into a limit cycle about the target by the optimal feedback law for most non-origin target points. The period of this limit cycle is dependent on the position of the target point in state-space. We characterize the time period of the limit cycles thus created by the optimal control strategy. A preliminary version of this article was presented in [10].

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2. Preliminaries and analysis

We consider a linear time invariant system with n-dimensional state-space:

$$\dot{x}(t) = Ax(t) + bu(t); \quad x(0) = x_0 \tag{1}$$

where $x : [0, \infty) \to \mathbb{R}^n$ is the state vector and the input $u(t)$ is a measurable function $u : [0, \infty) \to \mathbb{R}$ which belongs to a set of admissible inputs $U = \{u \in L^\infty[0, \infty) : |u(t)| \le 1 \text{ a.e. } t \in [0, \infty)\}$. We assume that the eigenvalues of matrix A are distinct and rational and the pair (A, b) is controllable. Let us denote by X_p the set of initial conditions from which system (1) can be transferred to $p \in \mathbb{R}^n$ in some time $t > 0$. We assume access to state-variables and compute the time optimal feedback law $h : X_p \to [-1, 1]$ to transfer any initial condition $x_0 \in X_p$ to the target point p. The eigenvalues of A are assumed to be distinct and rational. The *reachable set at time* $t > 0$ (denoted by $R_p(t)$) to point p is the set of all the points $x \in \mathbb{R}^n$ that can be driven to p in time t using the admissible control $u(t) \in U$ [16].

$$R_p(t) = \left\{ e^{-At}p - \int_0^t e^{-A\tau} bu(\tau)d\tau : u(t) \in U \right\}. \tag{2}$$

The set of all the states that can be driven to p using $u(t) \in U$ is called the *reachable set* to the point p and is $X_p = \bigcup_{t\in[0,\infty)} R_p(t)$. When p is the origin, we get the set of initial conditions $R_0(t)$ that can be driven to the origin with input $u(t) \in U$ in time t : $R_0(t) = \left\{ \int_0^t e^{-A\tau} bu(\tau)d\tau : u(t) \in U \right\}$. Then, the *set of null-controllable states* [12] is given by $X_0 = \bigcup_{t\in[0,\infty)} R_0(t)$. From point p the *attainable set* at time t is the set of all the states that can be reached from the point p using admissible control $u(t) \in U$ in time $t > 0$ [16].

$$\mathcal{A}_p(t) = \left\{ e^{At}p + \int_0^t e^{A(t-\tau)} bu(\tau)d\tau : u(t) \in U \right\}. \tag{3}$$

Taking the union of set (3) for all time $t > 0$ we get the *attainable set* from the point p which is $\mathcal{A}_p = \bigcup_{t\in[0,\infty)} \mathcal{A}_p(t)$ [16].

A point $p \in \mathbb{R}^n$ is said to be *a constrained controllable* for system (1), if it is in the interior of X_p i.e. $p \in \text{int}(X_p)$. In other words all points in the neighborhood of a constrained controllable point p must be transferable to p. We restrict ourselves by considering the target states which are constrained controllable points because of following reason. If a point p is not a constrained controllable point, then for some initial conditions in the neighborhood of p it is impossible to *transfer* the state-trajectory to target point p. Thus, such a target point is not useful for practical purposes. Hence-forth we assume p to be a constrained controllable point.

Next we review the notion of switching surfaces, which will be used subsequently to construct the time optimal feedback law. It is well known that the optimal control for minimum time state-transfer is bang–bang with at most $n-1$ switches [4]. Let $M^+_{p,k}$ be the set of initial conditions which can be steered to the target p using bang–bang input with at most $k-1$ switches and initial input $u = 1$. To characterize the set $M^+_{p,k}$, we first define the set $V_k := \{(t_1, t_2, \dots, t_k) : 0 \le t_1 \le t_2 \le \dots \le t_k < \infty\}$ for $k = 1, \dots, n$. Next, we use (2) to define functions $F^+_{p,k}$ and $F^-_{p,k}$ as follows.

$$F^\pm_{p,k} : V_k \to \mathbb{R}^n$$

$$F^\pm_{p,k} : (t_1, \dots, t_k) \mapsto e^{-At_k}p \pm \left(-\int_0^{t_1} + \dots + (-1)^k \int_{t_{k-1}}^{t_k} \right) e^{-A\tau} bd\tau. \tag{4}$$

Then, $M^+_{p,k} = \{x : x = F^+_{p,k}(v), \ \forall v \in V_k\}$ and similarly, $M^-_{p,k}$ is the set of such conditions with initial input $u = -1$ defined by $M^-_{p,k} = \{x : x = F^-_{p,k}(v), \ \forall v \in V_k\}$. Thus the set of all states that can be driven to p in at most $k-1$ switches is defined as

$$M_{p,k} = M^+_{p,k} \cup M^-_{p,k}. \tag{5}$$

The sets $M_{p,k}$ for $k = 1, \dots, n$ obey the inclusion relation $M_{p,0} \subset M_{p,1} \subset \dots \subset M_{p,n}$. From the existence and uniqueness of time optimal control to transfer any point in X_p to the target point p (see [4]), we get $M_{p,n} = X_p$. In other words, the reachable set to point p using only bang–bang input (with at most $n-1$ switches) is in fact the entire reachable set to p (i.e. X_p) and hence by (5), $X_p = M^+_{p,n} \cup M^-_{p,n}$.

3. Using $M_{p,k}$ for feedback

The nested sequence of sets $M_{p,k}$ $(k = 1, \dots, n-1)$ can be used to synthesize time optimal feedback control. Assume $p = 0$, temporarily, to simplify the following explanation. To drive the system from any state in X_0 to $p = 0$ in minimum time, the control value must ideally switch as follows. Consider $x_0 \in M^+_{0,n}$. Then initially, input $u = +1$ should be applied, which pushes x from $M^+_{0,n}$ to the manifold $M^-_{0,n-1}$. As soon as $x \in M^-_{0,n-1}$, the input should switch to $u = -1$, which then pushes x to $M^+_{0,n-2}$ and so on. Finally input at $(n-1)^{\text{th}}$ switch pushes x from $M^+_{0,1}$ (if n is odd) or $M^-_{0,1}$ (if n is even) to the target $p = 0$. Similar sequence is valid for $x_0 \in M^-_{p,n}$ with opposite signs of input [6]. This switching algorithm works ideally for $p = 0$, because the sets $M^+_{0,n} \setminus M^+_{0,n-1}$ and $M^-_{0,n} \setminus M^-_{0,n-1}$ are disjoint and $M^+_{0,n} \cap M^-_{0,n} = M_{0,n-1}$ is a set of measure zero in M_n (see [12]). Hence the initial input is determined uniquely from whether $x_0 \in M^+_{0,n}$ or $x_0 \in M^-_{0,n}$.

However, for most non-origin target points (i.e., $p \neq 0$), the set $M^+_{p,n} \cap M^-_{p,n}$ is of non-zero measure in $M_{p,n}$. This implies that points in the set $M^+_{p,n} \cap M^-_{p,n}$ can be driven to p by more than one, distinct bang–bang inputs with at most $n-1$ discontinuities. However, uniqueness of time-optimal control guarantees that only one of them is time-optimal. All points in the set X_p other than those which lie in the set $M^+_{p,n} \cap M^-_{p,n}$, belong to exactly one of the sets $M^+_{p,n}$ or $M^-_{p,n}$. All the initial conditions $x_0 \in X_p \setminus (M^+_{p,n} \cap M^-_{p,n})$ can be driven to p in minimum time by using the time optimal switching (exactly as in $p = 0$ case). The situation is demonstrated by an example next.

Example 1. For a second order LTI system (1) with $A = diag(1, 2)$, $b = [1 \quad 1]^T$ and $|u| \le 1$, the corresponding structure of X_p with $p = [0.33 \quad 0]^T$ is shown in Fig. 1. This figure illustrates that $M^+_{p,2} \cap M^-_{p,2}$ is non-empty. For an initial condition $x_0 \in M^-_{p,2} \setminus (M^+_{p,2} \cap M^-_{p,2})$ shown in the figure, input $u = -1$ steers the state-trajectory $x(t)$ towards $M^+_{p,1}$. As soon as $x(t) \in M^+_{p,1}$, the input $u = +1$ directs the state-trajectory towards p.

For initial conditions $x_0 \in M^+_{p,n} \cap M^-_{p,n}$, we have two choices of initial control namely $u = +1$ and $u = -1$. Only one of them drives the system to p in minimum time. This initial choice can be computed beforehand by using PMP. However after the initial input has been applied, all the future control inputs are computed as per the algorithm proposed. If choice of initial input goes wrong, although time-optimality will be lost, the target point will be achieved in finite time.

Before giving an algorithm to compute $M_{p,k}$, we note that, for any real similarity transformation $\widehat{x} = Tx$ on a system, the corresponding set $\widehat{M}^\pm_{p,k} = \{\widehat{x} = Tx : \ x \in M^\pm_{p,k}\}$ [12]. As system matrix A has distinct rational eigenvalues, we assume A to be diagonal without loss of generality.

4. Parametric representation

All $x \in M_{p,k}$ are characterized by the functions $F^+_{p,k}$ or $F^-_{p,k}$ (given by (4)) defined over V_k. Note that A is assumed to be diagonal.