Systems & Control Letters 79 (2015) 30–38

Contents lists available at ScienceDirect

Systems & Control Letters

journal homepage: www.elsevier.com/locate/sysconle

# Continuous dependence of optimal control to controlled domain of actuator for heat equation☆

Guojie Zheng [a,c], Bao-Zhu Guo [b,c,*], M. Montaz Ali [c]

[a] College of Mathematics and Information Science, Henan Normal University, Xinxiang, 453007, PR China
[b] Academy of Mathematics and Systems Science, Academia Sinica, Beijing 100190, PR China
[c] School of Computational and Applied Mathematics, University of the Witwatersrand, Wits 2050, Johannesburg, South Africa

ARTICLE INFO

*Article history:*
Received 2 November 2014
Received in revised form
30 December 2014
Accepted 19 February 2015
Available online 28 March 2015

*Keywords:*
Heat equation
Time optimal control
Norm optimal control
Controlled domain
Continuous property

ABSTRACT

In this paper, we consider continuous dependence of the optimal control with respect to the actuator domain which is varying as open subset in the spatial domain for a multi-dimensional heat equation. Both time optimal control and norm optimal control problems are considered. The reason behind combining these two problems together is that these two problems are actually equivalent: The energy to be used to drive the system to target set in minimal time interval is actually the minimal energy of driving the system to target set in this minimal time interval, and visa versa. It is shown that both optimal control and optimal cost are continuous with respect to open controlled actuator domain under the Lebesgue measure.


## 1. Introduction

From the shape optimization point of view, the location of actuator can be regarded as a controller for the systems described by partial differential equations (PDEs). There are many works seeking optimal location of the optimal controls for PDEs, for instance, [1–5]. In this paper, however, we study continuous property of the optimal control with respect to the controlled actuator domain for a multi-dimension heat equation, which is different from continuous dependence on the initial value studied in [6–9].

Suppose that $\Omega \subset \mathbb{R}^d$ is a bounded domain with $C^\infty$ boundary. Let $\omega \subset \Omega$ be a nonempty open subset which is considered to be the controlled actuator domain. Denote by $\chi_\omega$ the characteristic function of $\omega$. In this paper, we are concerned with the following controlled multi-dimensional heat equation:

$$\begin{cases} \partial_t y(x,t) - \triangle y(x,t) = \chi_\omega u(x,t) & \text{in } \Omega \times (0,+\infty), \\ y(x,t) = 0 & \text{on } \partial\Omega \times (0,+\infty), \\ y(x,0) = y_0(x) & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $y_0 \in L^2(\Omega)$ is the initial state, and $u \in L^\infty(0,\infty; L^2(\Omega))$ is the control input. We consider system (1.1) in the state space $L^2(\Omega)$ with the usual norm and inner product denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively, without specific explanation in what follows. The unique solution of (1.1) is denoted by $y_\omega(\cdot; y_0, u)$ to represent the relation of the solution with the control input $u$, the actuator domain $\omega$, and the initial state $y_0$.

For a given fixed $M > 0$, the constraint control set is taken as

$$\mathcal{U}_M = \{v \in L^\infty(0,+\infty; L^2(\Omega)) \mid \|v(t)\|_{L^2(\omega)} \le M, \\ \forall\, t \in [0,+\infty) \text{ a.e.}\}$$

Let $r > 0$ be a given positive number. The target set is chosen as the closed ball $B(0,r) \equiv \{w \in L^2(\Omega) | \|w\| \le r\}$ and suppose that $y_0 \notin B(0,r)$. The time optimal control problem studied in this paper reads as follows:

$$(\mathcal{TP}_{\omega,M}): \min_{u\in\mathcal{U}_M} \{T | y_\omega(T; y_0, u) \in B(0,r)\}.$$

The time $T_{\omega,M}(y_0)$ is called the optimal time if $T_{\omega,M}(y_0) \equiv \min_{u\in\mathcal{U}_M}\{T|y(T; y_0, \chi_\omega u) \in B(0,r)\}$, while a control $u_{\omega,M} \in \mathcal{U}_M$ satisfying $y_\omega(T_{\omega,M}(y_0); y_0, u_{\omega,M}) \in B(0,r)$ is called an optimal

☆ This work was partially supported by the National Research Foundation of South Africa under Grant CPR2010030300009918, the National Natural Science Foundation of China (U1204105, 61203293), and the National Basic Research Program of China (2011CB808002).
* Corresponding author at: Academy of Mathematics and Systems Science, Academia Sinica, Beijing 100190, PR China.
*E-mail addresses:* guojiezheng@yeah.net (G. Zheng), bzguo@iss.ac.cn (B.-Z. Guo).

http://dx.doi.org/10.1016/j.sysconle.2015.02.007
0167-6911/

control. Clearly, $T_{\omega,M}(\cdot)$ defines a functional from $L^2(\Omega)$ to $\mathbb{R}^+ \equiv [0, +\infty)$. For notational simplicity, we just write $T_{\omega,M}$ as the optimal time for problem $(\mathcal{TP}_{\omega,M})$ if there is no risk to make any confusion.

There are many researches concerning time optimal control problem, see, for instance, [10–14], name just a few.

**Remark 1.1.** When $M = 0$, $T_{\omega,0}$ is just the minimal time for the solution of Eq. (1.1) to enter the target set $B(0, r)$ without control. For convenience, we regard $u_{\omega,M}(x, t) = 0$ when $x \notin \omega$ and $t \in (0, +\infty)$, or $t > T_{\omega,M}$ and $x \in \Omega$.

From the shape optimization point of view, the $\omega$ can be regarded as a controller as well for problem $(\mathcal{TP}_{\omega,M})$. The objective of this paper is to investigate the continuous dependence of optimal control with the variation of actuator domain. To this purpose, we need to define a metric-like relation among open actuator domains. For any measurable set $A \subseteq \mathbb{R}^d$, let $m(A)$ be its Lebesgue measure in $\mathbb{R}^d$. Consider the set

$$\mathcal{A} = \{A \mid A \text{ is a nonempty open subset of } \Omega\}.$$

We can define a positive function over $\mathcal{A}$ as:

$$d(A_1, A_2) = m[(A_1 \setminus A_2) \cup (A_2 \setminus A_1)], \quad \forall A_1, A_2 \in \mathcal{A}. \tag{1.2}$$

As seen in Appendix, $d(\cdot, \cdot)$ can be indeed regarded as a metric between open sets when we limit the open sets to be convex. The main results of this paper are stated as Theorems 1.1 and 1.2.

**Theorem 1.1.** *Let $\{\omega_n\}_{n=1}^\infty \subset \mathcal{A}$ satisfy $d(\omega_n, \omega) \to 0$. Let $T_{\omega,M}$ and $u_{\omega,M}$ be the optimal time and the optimal control to problem $(\mathcal{TP}_{\omega,M})$, and let $T_{\omega_n,M}$ and $u_{\omega_n,M}$ be the optimal time and the optimal control to problem $(\mathcal{TP}_{\omega_n,M})$, respectively. Then,*

$$T_{\omega_n,M} \to T_{\omega,M} \quad \text{as } n \to \infty, \tag{1.3}$$

$$u_{\omega_n,M} \to u_{\omega,M} \quad \text{strongly in } L^2(0, T_{\omega,M}, ; L^2(\Omega)) \text{ as } n \to \infty. \tag{1.4}$$

*Moreover, if $\{\omega_n\}_{n=1}^\infty \subset \mathcal{A}$ satisfies the following condition* (C)*:*

(C)*: There exist $x_0 \in \mathbb{R}^d$, positive number $l > 0$, and $N \in \mathbb{N}$ such that $\bigcap_{n=N}^\infty \omega_n \supseteq B_l \equiv \{x \in \mathbb{R}^d \mid \|x - x_0\|_{\mathbb{R}^d} \le l\}$.*

*Then for any $\eta \in (0, T_{\omega,M})$,*

$$u_{\omega_n,M} \to u_{\omega,M} \quad \text{strongly in } L^\infty(0, T_{\omega,M} - \eta, ; L^2(\Omega)) \text{ as } n \to \infty. \tag{1.5}$$

**Remark 1.2.** (i) The Assumption (C) holds when we limit the open sets to be convex (see Appendix).

(ii) If the target set is zero instead of ball $B(0, r)$ as discussed in this paper, we need a uniform constant in "observability inequality" for the dual system of null controllable system (1.1) to guarantee convergence of the optimal times [15]. Precisely, for given $T > 0$, there exists $C_T > 0$ which is independent of $n$ such that for any $y_0 \in L^2(\Omega)$ and any $n \in \mathbb{N}$,

$$C_T \|\varphi(0)\|^2_{L^2(\Omega)} \le \int_0^T \|\chi_{\omega_n}\varphi(t)\|^2_{L^2(\Omega)} dt, \tag{1.6}$$

where $\varphi(t)$ is the solution of the following dual system of (1.1):

$$\begin{cases} \partial_t\varphi(x, t) + \triangle\varphi(x, t) = 0 & \text{in } \Omega \times (0, T), \\ \varphi(x, t) = 0 & \text{on } \partial\Omega \times (0, T), \\ \varphi(x, T) = \varphi_0(x) & \text{in } \Omega. \end{cases}$$

It can be shown that the Assumption (C) implies condition (1.6). However, we do not need Assumption (C) until (1.5). In other words, we do not need the condition (1.6) for conclusions (1.3)–(1.4) when the target set is a ball $B(0, r)$ whereas the condition (1.6) is necessary for single zero target set even for the conclusions (1.3)–(1.4). The same remark is applied to Theorem 1.2.

The aforementioned results on the time optimal control can be generalized to the norm optimal control. To this purpose, we first rewrite (1.1) by replacing control $u$ with $f$ for the following norm optimal control problem:

$$\begin{cases} \partial_t y(x, t) - \triangle y(x, t) = \chi_\omega f(x, t) & \text{in } \Omega \times (0, T), \\ y(x, t) = 0 & \text{on } \partial\Omega \times (0, T), \\ y(x, 0) = y_0(x) & \text{in } \Omega, \end{cases} \tag{1.7}$$

where $\omega$, $\chi_\omega$, and $\Omega$ are the same as aforementioned, and $T > 0$ is a given number.

We also consider the norm optimal control problem for system (1.7) in the state space $L^2(\Omega)$. Denote, by abuse of notation, by $y_\omega(\cdot; y_0, f)$ the solution of Eq. (1.7) associated with the control $f$, the actuator domain $\omega$, and the initial state $y_0$. The norm optimal control problem that we are concerned with can be described as follows:

$$(\mathcal{NP}_{\omega,T}) : \inf_{f\in\mathcal{F}_T} \|f\|_{L^\infty(0,T;L^2(\Omega))},$$

where the admissible control set $\mathcal{F}_T$ is defined by

$$\mathcal{F}_T = \{f \in L^\infty(0, T; L^2(\Omega)) | y_\omega(T; y_0, f) \in B(0, r)\}. \tag{1.8}$$

By null controllability of system (1.7) (see e.g., [15]), $\mathcal{F}_T$ is not empty for any $T > 0$. The admissible control $f \in \mathcal{F}_T$ is denoted by $f_{\omega,T}$ to represent the relation of $f$ with actuator domain $\omega$ and time $T$. The number $M$ is called the optimal norm if $M = \inf_{f\in\mathcal{F}_T} \|f\|_{L^\infty(0,T;L^2(\Omega))}$, and $f_{\omega,T} \in \mathcal{F}_T$ is called an optimal control if $\|f_{\omega,T}\| = M$.

**Theorem 1.2.** *Suppose that $\{\omega_n\}_{n=1}^\infty \subset \mathcal{A}$ satisfies $d(\omega_n, \omega) \to 0$. Let $f_{\omega,T}$ and $f_{\omega_n,T}$ be the optimal controls to problems $(\mathcal{NP}_{\omega,T})$ and $(\mathcal{NP}_{\omega_n,T})$, respectively. Then*

$$\|f_{\omega_n,T}\|_{L^\infty(0,T;L^2(\Omega))} \to \|f_{\omega,T}\|_{L^\infty(0,T;L^2(\Omega))} \quad \text{as } n \to \infty, \tag{1.9}$$

*and*

$$y_{\omega_n}(T; y_0, f_{\omega_n,T}) \to y_\omega(T; y_0, f_{\omega,T}) \quad \text{strongly in } L^2(\Omega) \text{ as } n \to \infty. \tag{1.10}$$

*Moreover, if $\{\omega_n\}_{n=1}^\infty \subset \mathcal{A}$ satisfies the following condition* (C)*:*

(C)*: There exist $x_0 \in \mathbb{R}^d$, positive number $l > 0$, and $N \in \mathbb{N}$ such that $\bigcap_{n=N}^\infty \omega_n \supseteq B_l \equiv \{x \in \mathbb{R}^d \mid \|x - x_0\|_{\mathbb{R}^d} \le l\}$.*

*Then for any $\eta \in (0, T)$,*

$$f_{\omega_n,T} \to f_{\omega,T} \quad \text{strongly in } L^\infty(0, T - \eta, ; L^2(\Omega)) \text{ as } n \to \infty. \tag{1.11}$$

The Problems $(\mathcal{TP}_{\omega,M})$ and $(\mathcal{NP}_{\omega,T})$ are closely related. An interesting result of [15] says that these two problems are actually equivalent: The energy to be used to drive the system to zero in minimal time interval is actually the minimal energy of driving the system to zero in this minimal time interval, and visa versa. The main results stated above can be regarded as a sensitivity or stability of the optimal control pairs and optimal costs for problems $(\mathcal{TP}_{\omega,M})$ and $(\mathcal{NP}_{\omega,T})$. The motivation of this study is apparent. From the perspective of applications, if the domain $\omega$ is also regarded as a controller for system (1.1) or (1.7), then such a control corresponds to the actuator domain of the original control, which plays a key role in industry applications for temperature control. The choice of location for the actuator is a distinctive problem for systems described by the partial differential equations (see e.g., [16]). In this regard, there appears perturbation problem for actuator domains. The study of the variation of actuator domains on influence of the control performance therefore becomes important. Among these performances, optimal control and optimal cost are the major concerns.