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Reducing variables in Model Predictive Control of linear system with disturbances using singular value decomposition

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ABSTRACT

Arising from the need to reduce online computations of Model Predictive Controller, this paper proposes an approach for a linear system with bounded additive disturbance using fewer variables than the standard. The new variables are chosen so that they transfer the maximal energy to the control inputs. Several other features are introduced. These include an auxiliary state to ensure recursive feasibility, an initialization procedure that recovers a substantial portion of the original domain of attraction arising from the use of fewer variables. A comparison of the domains of attraction associated with the new variables is also discussed. Run-time computational advantage of more than an order of magnitude compared with standard approach is demonstrated using several numerical examples although a more expensive initialization is needed.

1. Introduction

This paper considers the Model Predictive Control (MPC) framework for the discrete-time system

$$x(t+1) = Ax(t) + Bu(t) + w(t), \quad w(t) \in W, \tag{1}$$

$$x(t) \in X \subset \mathbb{R}^n, \qquad u(t) \in U \subset \mathbb{R}^m \tag{2}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $w \in \mathbb{R}^n$ are the state, control and disturbance of the system and X and U are appropriate constraint sets on x and u respectively. It focuses on the reduction of online computations using fewer variables than the standard. Past research in this direction include Generalized Predictive control [1], blocking parametrization [2–6], various parametrizations [7,8] and others. The use of fewer variables naturally leads to a lower online computational effort but it also results in a loss in robustness against unexpected disturbances in the form of a smaller domain of attraction, an issue that is seldom discussed in the literature. Such issues, together with recursive feasibility, closed-loop stability, loss in performance and the choice of the variables remains open research issues in this line of work. This paper is an attempt to address some of these issues by proposing the choice of new variables based on Singular Value Decomposition (SVD), introducing an auxiliary state for recursive feasibility and a procedure that computes the initial auxiliary state so as to recover a substantial portion of the original domain of attraction. The proposed approach has minimal computational load during run time. It does require more computations for the initial auxiliary state. Since this requirement is only needed at initialization, it is not too restrictive.

The use of SVD in MPC is not new [9,10]. In [9], the reduced variables are such that loss in the performance index, as compared to the standard, is minimal. While reasonable, its effectiveness is unclear when the states are far from the origin where maneuvering the constraints is the primary concern. The approach of [10] performs the SVD of a matrix consisting of the snapshots of past control inputs. Implicit in this approach is that past inputs are good representations of future maneuvers. Unlike them, this work uses a reduced set of variables that contains the maximal amount of energy, a statement that will be made precise in Section 3.

The notations used in this paper are standard. Non-negative and positive integer sets are indicated by $\mathbb{Z}_0^+$ and $\mathbb{Z}^+$ respectively with $\mathbb{Z}_i^k := \{i, i+1, \ldots, k\}$, $k \geq i$. Similarly, $\mathbb{R}_0^+$ and $\mathbb{R}^+$ refer respectively to the sets of non-negative and positive real numbers. Given a matrix A and a vector b, $A_{:j}$, $A_{i:}$ are respectively the jth column and ith row of A and b_k is the kth element. For a square matrix Q, $Q \succ (\succeq) 0$ means Q is positive definite (semi-definite). For any set $X,\ Y \subset \mathbb{R}^n$, $X \oplus Y := \{x + y : x \in X, y \in Y\}$ is the Minkowski sum of X and Y and $X \ominus Y := \{z : z + y \in X, \forall y \in Y\}$ is the Pontryagin difference of the two sets. Vector 2-norm and matrix induced 2-norm are denoted by the standard $\|\cdot\|$ notation with $\|x\|_Q^2 = x^T Q x$ for $Q \succ 0$. For distinction from the true system state $x(t)$ and control $u(t)$, the kth predicted state and control are represented as x_k, u_k respectively, or as $x_{k|t}$ and $u_{k|t}$ where the reference to time is needed. Hence, $x_{0|t} = x(t)$ and $u_{0|t} = u(t)$. I_n is and $n \times n$ identity matrix.

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The rest of this paper is organized as follows. Section 2 states the assumptions needed and reviews past results that handles the additive disturbance. The finite horizon optimization problem is stated in full in Section 3 including the choice of the new variables and the auxiliary states. Feasibility and stability results are stated in Section 4. Approach to recover the domain of attraction via an initialization process is provided in Section 5. Numerical results and the conclusions are the remaining two sections.

2. Preliminaries

This section reviews known results of MPC for system (1) and (2), sets up the notations for subsequent discussion and begins with the assumptions needed. (**A1**): (A, B) is stabilizable and x is measurable. (**A2**): W is polytope that contains the origin in its interior (**A3**): Constraint sets $X \subset R^n$ and $U \subset R^m$ are polytopes that contain the origin in their respective interiors. (**A4**): A set $X_f \subset \mathbb{R}^n$ and a feedback gain $K \in \mathbb{R}^{m\times n}$ are determined such that X_f is a constraint-admissible disturbance invariant set for system (1) with $u = Kx$.

Assumption (A1) is a standard requirement of the system. (A2) and (A3) are mild assumptions on the disturbance and constraint sets, made out of computational requirement. (A4) refers to the existence of X_f such that

$$A_K x + w \in X_f, \qquad Kx \in U \quad \text{for all } x \in X_f \text{ and for all } w \in W \tag{3}$$

where $A_K := A+BK$ is Schur stable. Such a X_f is known to exist [11] under assumptions (A2), (A3) for sufficiently small W and can be computed via an iterative procedure that terminates in a finite number of steps. Define

$$\begin{aligned} &F_0 := \{0\}, \qquad F_t := W \oplus A_K W \oplus \cdots \oplus A_K^{t-1} W, \\ &F_\infty := \lim_{j\to\infty} F_j. \end{aligned} \tag{4}$$

System (1) with $u = Kx$ becomes $x(t+1) = A_K x(t) + w(t)$ and its state are

$$x(t) = A_K^t x(0) + f_t, \quad f_t = \sum_{j=0}^{t-1} A_K^{t-1-j} w(j) \tag{5}$$

with $f_t \in F_t$ for $t \in \mathbb{Z}_0^+$. Also, F_∞ is the minimal invariant set for $x(t+1) = A_K x(t) + w(t)$ with the property that $A_K F_\infty \oplus W \subset F_\infty$. Additional properties and computations of F_∞ are discussed in [12, 13]. With these definitions, an additional assumption, (**A5**): $F_\infty \subset int(X_f)$ is needed whose satisfaction can be achieved when the size of W is sufficiently small.

As disturbances are present in (1), it is known that the search for an admissible control in MPC should be over some family of feedback policies instead of the direct values of $\{u_0, u_1, \ldots, u_{N-1}\}$ [14]. For this purpose, the control parametrization used here is $u_i = Kx_i + d_i$ where K is fixed and given. Consider (1) in the form of $x_{i+1} = Ax_i + Bu_i + w_i$ and the nominal (disturbance-free) system of (1) $\bar{x}_{i+1} = A\bar{x}_i + B\bar{u}_i$ with

$$\bar{u}_i = K\bar{x}_i + d_i. \tag{6}$$

It can be shown that $x_i - \bar{x}_i = f_i$ and $u_i = \bar{u}_i + Kf_i$ for any $i \in \mathbb{Z}^+$ where f_i is that given by (5). Hence, the nominal system allows for a simpler MPC treatment by appropriately strengthening the constraints [15–17]. Consider a MPC horizon length of $N \in \mathbb{Z}^+$ and let

$$\bar{X}_f := X_f \ominus F_N, \qquad X_i := X \ominus F_i, \qquad U_i := U \ominus KF_i. \tag{7}$$

It follows from $x_i = \bar{x}_i + f_i$ and $u_i = \bar{u}_i + Kf_i$ that

$$\bar{x}_i \in X_i, \qquad \bar{u}_i \in U_i \Longrightarrow x_i \in X, \quad u_i \in U \quad \forall i \in \mathbb{Z}_0^{N-1} \tag{8}$$

$$\bar{x}_N \in \bar{X}_f \Longrightarrow x_N \in X_f. \tag{9}$$

Eq. (8) shows that constraints (2) can be replaced by $\bar{x}_i \in X_i$, $\bar{u}_i \in U_i$ which, in turn, implies that MPC problem for (1) can be formulated using $(\bar{x}_i, \bar{u}_i)$ as the state and control variables with appropriate constraints. Doing so, expressions of (1) and (6) can be equivalently stated as

$$\begin{aligned} \bar{\boldsymbol{x}}^+ &= \mathbf{A}\bar{\boldsymbol{x}} + \mathbf{B}\bar{\boldsymbol{u}} := \begin{pmatrix} A & 0 & \cdots & 0 \\ 0 & & \ddots & 0 \\ 0 & \cdots & 0 & A \end{pmatrix} \begin{pmatrix} \bar{x}_0 \\ \vdots \\ \bar{x}_{N-1} \end{pmatrix} \\ &\quad + \begin{pmatrix} B & 0 & \cdots & 0 \\ 0 & & \ddots & 0 \\ 0 & \cdots & 0 & B \end{pmatrix} \begin{pmatrix} \bar{u}_0 \\ \vdots \\ \bar{u}_{N-1} \end{pmatrix} \end{aligned} \tag{10}$$

$$\bar{\boldsymbol{u}} = \mathbf{K}\bar{\boldsymbol{x}} + \boldsymbol{d} := \begin{pmatrix} K & 0 & \cdots & 0 \\ 0 & & \ddots & 0 \\ 0 & \cdots & 0 & K \end{pmatrix} \begin{pmatrix} \bar{x}_0 \\ \vdots \\ \bar{x}_{N-1} \end{pmatrix} + \begin{pmatrix} d_0 \\ \vdots \\ d_{N-1} \end{pmatrix} \tag{11}$$

where $\bar{\boldsymbol{x}}^+ := [\bar{x}_1^T \cdots \bar{x}_N^T]^T \in \mathbb{R}^{Nn}$, $\bar{\boldsymbol{x}}^T := [\bar{x}_0^T \cdots \bar{x}_{N-1}^T] \in \mathbb{R}^{Nn}$, $\bar{\boldsymbol{u}}^T := [\bar{u}_0^T \cdots \bar{u}_{N-1}^T] \in \mathbb{R}^{Nm}$ and $\boldsymbol{d}^T := [d_0^T \cdots d_{N-1}^T] \in \mathbb{R}^{Nm}$. Using (5), these two equations can be equivalently stated as

$$\begin{aligned} \bar{\boldsymbol{x}}^+ &= \mathbf{A}_K\bar{x}_0 + \mathbf{B}_K\boldsymbol{d} := \begin{pmatrix} A_K \\ \vdots \\ A_K^N \end{pmatrix} \bar{x}_0 \\ &\quad + \begin{pmatrix} B & 0 & \cdots & 0 \\ A_K B & B & \cdots & 0 \\ \vdots & & & \\ A_K^{N-1}B & A_K^{N-2}B & \cdots & B \end{pmatrix} \boldsymbol{d}, \end{aligned} \tag{12}$$

$$\begin{aligned} \bar{\boldsymbol{u}} &= \mathbf{K}_A\bar{x}_0 + \mathbf{G}\boldsymbol{d} := \begin{pmatrix} K \\ KA_K \\ \vdots \\ KA_K^{N-1} \end{pmatrix} \bar{x}_0 \\ &\quad + \begin{pmatrix} I & 0 & \cdots & 0 \\ KB & I & & 0 \\ \vdots & \vdots & & \\ KA_K^{N-2}B & \cdots & KB & I \end{pmatrix} \boldsymbol{d} \end{aligned} \tag{13}$$

where $\mathbf{A}$, $\mathbf{K}$, $\mathbf{B}$, $\mathbf{A}_K$, $\mathbf{K}_A$, $\mathbf{B}_K$ and $\mathbf{G}$ are implicitly defined by the equations above. For notational convenience, let

$$\begin{aligned} C &:= C(X_0, \ldots, X_{N-1}, U_0, \ldots, U_{N-1}, \bar{X}_f) \\ &= \{(\bar{\boldsymbol{u}}, \bar{\boldsymbol{x}}, \bar{x}_N) : \bar{x}_i \in X_i, \bar{u}_i \in U_i, i \in \mathbb{Z}_0^{N-1} \text{ and } \bar{x}_N \in \bar{X}_f\} \end{aligned} \tag{14}$$

where X_i, U_i, $\bar{X}_f$ are those defined in (7). Note that under (A2)–(A4), X_i, U_i and $\bar{X}_f$ are polytopes represented by linear inequalities in $\mathbb{R}^n$, $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively and C is a polytope in $\mathbb{R}^{Nm+(N+1)n}$. The N-stage MPC controller solves at every t, with parameter $x = x(t)$ the following optimization problem:

$$\mathcal{P}_d(x) : \min_{\boldsymbol{d}}\{J_s(\boldsymbol{d}) : \bar{x}_0 = x, (12), (13), (\bar{\boldsymbol{u}}, \bar{\boldsymbol{x}}, \bar{x}_N) \in C\} \tag{15}$$

where $J_s(\boldsymbol{d})$ is some appropriate convex function of $\boldsymbol{d}$. Since the constraints are all linear equations or inequalities in $\boldsymbol{d}$, $\mathcal{P}_d(x)$ is a convex programming problem. $\mathcal{P}_d(x)$ can also be equivalently stated using (10), (11) instead of (12), (13) in (15). As a reference for comparison, the corresponding domain of attraction is

$$\mathcal{D}_d := \{x : \exists\, \boldsymbol{d} \text{ s.t. } \mathcal{P}_d(x) \text{ has a feasible solution}\}. \tag{16}$$

The cost function $J_s(\boldsymbol{d})$ is obtained from the standard Linear Quadratic (LQ) cost which takes the form

$$\sum_{i=0}^{N-1}[x_i^T Q x_i + u_i^T R u_i] + x_N^T P x_N \tag{17}$$