Systems & Control Letters 61 (2012) 1–5

Contents lists available at SciVerse ScienceDirect

Systems & Control Letters

journal homepage: www.elsevier.com/locate/sysconle

# Stability and stabilization of Boolean networks with impulsive effects

Fangfei Li, Jitao Sun *

*Department of Mathematics, Tongji University, Shanghai, China*

ARTICLE INFO

*Article history:*
Received 25 December 2010
Received in revised form
21 September 2011
Accepted 28 September 2011
Available online 20 November 2011

*Keywords:*
Boolean networks
Impulsive effects
Systems biology
Stability
Stabilization
Semi-tensor product of matrices

ABSTRACT

This paper investigates the stability and stabilization of Boolean networks with impulsive effects. After giving a survey on semi-tensor product of matrices, we convert a Boolean network with impulsive effects into impulsive discrete-time dynamics. Then, some necessary and sufficient conditions are given for the stability and stabilization of Boolean networks with impulsive effects. Finally, examples are provided to illustrate the efficiency of the obtained results.



## 1. Introduction

Systematic analysis of biological systems is becoming more and more important. A new view of biology, called systems biology is emerging. One of the major goals of systems biology is to develop a control theory for complex biological systems. Systems biology does not investigate individual genes, proteins or cells, one at a time. Rather, it studies the behavior and relationship of all of the cells, proteins, DNA and RNA in a biological systems, called the cellular networks.

In order to develop control strategies, we need a mathematical model for the biological systems. Boolean networks, introduced first by Kauffman [1], and then developed by Shmulevich et al. [2], Farrow et al. [3], Albert and Barabási [4] and many others, have became a powerful tool in describing, analyzing and simulating the cellular networks. In Boolean networks, the state of a gene can be described by a Boolean variable: active (1) or inactive (0). Moreover, interactions between the states of each gene can be determined by Boolean functions, which calculate the state of a gene from the activation of other genes. As Boolean networks can represent important features of living organisms, there are several methods developed for analyzing Boolean networks; see e.g. [5–9] for some interesting and recent developments on the topological structure analysis.

Recently, [10] introduces a new method, which converts a Boolean network into a conventional discrete-time dynamic system. Then some of the tools developed in control theory for analysis and control of dynamic systems become applicable. Cheng and Qi [11] and Zhao et al. [12] use this new technique to analyze the controllability and observability of Boolean control networks. Cheng et al. [13] investigates the realization, particularly, minimum realization of Boolean control networks. Li and Sun [14] studies the controllability of Boolean networks with time delays.

However, in the real world, many evolutionary processes especially the biological networks may experience abrupt changes of states at certain time instants. These maybe due to changes in the interconnections of subsystems, sudden environment changes, etc. These abrupt changes of states may occur at prescribed time instants and/or triggered by specified events along a particular trajectory. To describe mathematically an evolution of a real process with a short-term perturbation, it is sometimes convenient to neglect the duration of the perturbation and to consider these perturbations to be "instantaneous", that is in the form of impulse. It is known that many biological phenomena exhibit impulsive effects, such as Kruger–Theimer model for drug distribution and so on. There is a great deal of literature studying the systems with impulsive effects; for example, see [15,16]. Boolean networks are used widely in systems biology, but to the best of our knowledge, there have been no results to consider a Boolean network whose states experience impulsive effects at some time instants. In fact, Boolean networks involving impulsive effects, appear as a natural description of observed evolution phenomena of several real world problems. To study a Boolean network with impulsive effects is meaningful and challenging.

* Corresponding author. Tel.: +86 21 65983241 1307; fax: +86 21 65981985.
*E-mail address:* sunjt@sh163.net (J. Sun).

0167-6911/$ – see front matter © 2011 Elsevier B.V. All rights reserved.
doi:10.1016/j.sysconle.2011.09.019

As we know, the study of stability is an important motif, there have been many results about the stability and stabilization of dynamics; see e.g. [17–19]. Stability and stabilization analysis of Boolean networks is an interesting topic too. But because we are short of systematic tools to deal with a logic system, the results of the stability and stabilization of Boolean networks are few, for example, see [20] in the literature. To the best of our knowledge, there has been no result about the stability and stabilization of Boolean networks with impulsive effects. Motivated by the above, in this paper, we study the stability and stabilization of Boolean networks with impulsive effects.

The rest of this paper is organized as follows. Section 2 gives a brief review for the semi-tensor product of matrices and some notations. Vector form of Boolean variable and its basic properties are also discussed. In Section 3, we first use the tools introduced in Section 2 to convert a Boolean network with impulsive effects into impulsive discrete-time systems. Then, the stability and stabilization are discussed respectively. Necessary and sufficient conditions for global stability are proved. Meanwhile, the stabilization design via two types of controls for a Boolean network with impulsive effects are provided. One kind of control is open-loop, which is free Boolean sequence. Another kind of control is closed-loop. The efficiency of the results are demonstrated through examples in Section 4. Finally, concluding remarks are given in Section 5.

## 2. Preliminaries

In this section, we will introduce the semi-tensor product of matrices and the matrix expression of logic briefly, which are recapitulation results from [10].

### 2.1. Semi-tensor product of matrices

In this paper, the matrix product we use is semi-tensor product (STP) of matrices.

**Definition 1** (*[10]*)**.** 1. Let $X$ be a row vector of dimension $np$, and $Y$ be a column vector of dimension $p$. Then we split $X$ into $p$ equal-size blocks as $X^1, \dots, X^p$, which are $1 \times n$ rows. Define the STP, denoted by $\ltimes$, as

$$\begin{cases} X \ltimes Y = \sum_{i=1}^{p} X^i y_i \in R^n, \\ Y^T \ltimes X^T = \sum_{i=1}^{p} y_i (X^i)^T \in R^n. \end{cases}$$

2. Let $A \in M_{m\times n}$ and $B \in M_{p\times q}$. If either $n$ is a factor of $p$, say $nt = p$ and denote it as $A \prec_t B$, or $p$ is a factor of $n$, say $n = pt$ and denote it as $A \succ_t B$, then we define the STP of $A$ and $B$, denoted by $C = A \ltimes B$, as the following: $C$ consists of $m \times q$ blocks as $C = (C^{ij})$ and each block is

$$C^{ij} = A^i \ltimes B_j, \quad i = 1, \dots, m; j = 1, \dots, q,$$

where $A^i$ is the $i$-th row of $A$ and $B_j$ is the $j$-th column of $B$.

STP of matrices is a generalization of conventional matrix product. All the fundamental properties of conventional matrix product remain true. Based on this, we can omit the symbol $\ltimes$. There are also some basic properties of STP, for details, see [10].

### 2.2. Matrix expression of logic

In this subsection, we recall the matrix expression of logic. We give some notations first.

1. Define a delta set as $\Delta_k := \{\delta_k^i | i = 1, 2, \dots, k\}$, where $\delta_k^i$ is the $i_{\text{th}}$ column of the identity matrix $I_k$.

2. Assume a matrix $M = [\delta_n^{i_1}, \delta_n^{i_2}, \dots, \delta_n^{i_s}]$, we simply denote it as $M = \delta_n[i_1, i_2, \dots, i_s]$.

3. We denote the $i$th column of matrix $A$ by $\mathrm{Col}_i(A)$, and denote the set of columns of matrix $A$ by $\mathrm{Col}(A)$.

4. For a $n \times mn$ matrix $A$, we split it into $m$ square blocks as $\mathrm{Blk}_1(A), \mathrm{Blk}_2(A), \dots, \mathrm{Blk}_m(A)$. We denote the $i_{\text{th}}$ $n \times n$ blocks of $n \times mn$ matrix $A$ by $\mathrm{Blk}_i(A)$.

Second, we introduce the vector form of the logical variables. Let "1" and "0" represent the logical value "True" and "False" respectively, and $\mathscr{D} = \{1, 0\}$. We give logical values a vector form as: $T = 1 \sim \delta_2^1, F = 0 \sim \delta_2^2$. Then the logical variable $A(t)$ takes value from these two vectors, i.e.

$$A(t) \in \Delta := \Delta_2 = \{\delta_2^1, \delta_2^2\}.$$

The "$\mathscr{D}$" and "$\Delta$" will be used freely according to the variable types.

Next, we will introduce the logical matrix. A matrix $A \in M_{m\times n}$ is called a logical matrix, if the columns of $A$ are elements of $\Delta_m$. Denote the set of logical matrices by $\mathcal{L}$, the set of $n \times s$ logical matrices is defined by $\mathcal{L}_{n\times s}$.

Finally, we will introduce a lemma that will be used in our paper.

**Lemma 2** (*[10]*)**.** *Any logical function $L(A_1, \dots, A_n)$ with logical arguments $A_1, \dots, A_n \in \Delta$ can be expressed in a multi-linear form as*

$$L(A_1, \dots, A_n) = M_L A_1 A_2 \cdots A_n,$$

*where $M_L \in \mathcal{L}_{2\times 2^n}$ is unique, called the structure matrix of $L$.*

To see the definition of structure matrix and its concerning results, it refers to [10] for details.

## 3. Main results

### 3.1. Stability of Boolean networks with impulsive effects

A Boolean network with a set of Boolean variables $A_1, A_2, \dots, A_n$ can be described as

$$\begin{cases} A_1(t+1) = f_1(A_1(t),\ A_2(t), \dots, A_n(t)), \\ A_2(t+1) = f_2(A_1(t),\ A_2(t), \dots, A_n(t)), \\ \vdots \\ A_n(t+1) = f_n(A_1(t),\ A_2(t), \dots, A_n(t)), \end{cases} \tag{1}$$

where $A_1, \dots, A_n \in \mathscr{D}, f_i : \mathscr{D}^n \to \mathscr{D}, i = 1, 2, \dots, n$ are logical functions, $t = 0, 1, 2, \dots$.

In the Boolean networks, the set of Boolean variables $A_1, \dots, A_n$ represent proteins or other molecules. At each given time $t = 0, 1, 2, \dots$, a Boolean variable has only one of two different values 1 (active) or 0 (inactive). For example, see the Boolean model of cell growth, differentiation, and apoptosis (programmed cell death) introduced by Huang and Ingber [21], and the epigenetic model [22]. But in the real world, many evolutionary processes such as biological networks may experience abrupt changes of states at some time instants. Those sudden and sharp changes are often of very short duration and are assumed to occur instantaneously in the form of impulses. Boolean network is widely used in systems biology, it may be experience sudden changes of states too. This may be due to sudden environment changes and so on. Motivated by the above, in our paper, we consider the Boolean network (1) in the following case: the states $A_i$, $i = 1, 2, \dots, n$, experience sudden and sharp changes at time $t_k$, where $\{t_k\} \subseteq Z^+$, $0 = t_0 < t_1 < t_2 < \cdots < t_k < \cdots, k \in Z^+$. There are logical functions $g_i$, transfer the $A_i(t_k)$ into $g_i(A_1(t_k - 1), A_2(t_k -$