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# Lyapunov formulation of the large-scale, ISS cyclic-small-gain theorem: The discrete-time case[☆]

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ABSTRACT

This paper presents a Lyapunov formulation of the cyclic-small-gain theorem for dynamical networks composed of discrete-time input-to-state stable (ISS) subsystems. ISS-Lyapunov functions for dynamical networks satisfying the cyclic-small-gain condition are constructed from the ISS-Lyapunov functions of the subsystems.



## 1. Introduction

The small-gain theorem has been widely recognized as one of the most important tools for stability analysis and control design in both linear and nonlinear systems. Truly nonlinear small-gain theorems, i.e., those allowing general nonlinear gains, started with [1,2] using input–output operator theory and [3] relying upon Sontag's seminal work on input-to-state stability (ISS) [4,5] and its various equivalent characterizations by Sontag and Wang [6]. Early applications of nonlinear small-gain to control systems with saturation and dynamic uncertainties appear in [7,8], respectively. The discrete-time counterpart of continuous-time ISS and its small-gain theorem were developed later in [9,10]. Considering the relation between ISS and Lyapunov functions, the Lyapunov formulations of ISS small-gain theorems for continuous-time systems and discrete-time systems were reported in [11,12]. The Lyapunov-based small-gain results for hybrid systems have also been developed in [13–15].

Recently, some interesting extensions of the nonlinear small-gain theorem were obtained for general networks composed of ISS subsystems in [16–21] for both continuous-time and discrete-time systems. In [18], the interconnections in large-scale systems are formulated with nonlinear matrices and the small-gain condition was obtained in matrix form. In [20,21], the dynamical network with "max"-type interconnection was systematically studied and more general cyclic-small-gain criteria were developed for input-to-output stable (IOS) systems. The Lyapunov formulation of the matrix small-gain theorem for both continuous-time and discrete-time systems was developed in [22]. The Lyapunov formulation of cyclic-small-gain for continuous-time dynamical networks was reported in our recent work [23]. It should be mentioned that the "max"-type interconnection can also be represented with the notion of monotone aggregation functions in the recent work [22]. Refs. [24–26] show the equivalence of small-gain conditions and the existence of an asymptotically stable discrete-time comparison system induced by the interconnection gains of dynamical networks. In particular, Refs. [24,25] proved a vector small-gain theorem based on the relation between the vector gains in the general dynamical networks and the asymptotic stability of discrete-time systems with "MAX"-type system matrices.

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In view of the critical importance of discrete-time system theory in computer-aided control engineering applications [27], in this paper, we take a lead in generalizing the results of [23] to discrete-time dynamical networks. Interestingly, as we will see later, new phenomena arise; see Remarks 1 and 3 below. Also, some new technical lemmas are needed to achieve this generalization. Just like the continuous-time case, we can systematically construct a total ISS-Lyapunov function on the basis of ISS-Lyapunov functions of individual subsystems.

The rest of the paper is organized as follows. Section 2 gives some notations and definitions. Section 3 provides two ISS-Lyapunov formulations of the subsystems in the dynamical networks. Section 4 mainly studies the ISS-Lyapunov cyclic-small-gain for dynamical networks with subsystems formulated in dissipation form. In Section 5, we develop a counterpart of the results in Section 4 for dynamical networks with subsystems formulated in "gain margin" form. In Section 6, we employ an example to show the effectiveness of the main result. Section 7 presents some conclusions. The new technical lemmas are in Appendix.

## 2. Notations and definitions

Throughout the paper, we use $|x|$ to denote the Euclidean norm of $x \in \mathbb{R}^n$ and $x^T$ to denote the transpose of the vector $x$. We denote by $\mathbb{Z}_+$ the set of nonnegative integers.

A function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is *positive definite* if $\gamma(s) > 0$ for all $s > 0$ and $\gamma(0) = 0$. $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is *a class $\mathcal{K}$ function* if it is continuous, strictly increasing and $\gamma(0) = 0$; it is *a class $\mathcal{K}_\infty$ function* if it is a class $\mathcal{K}$ function and also satisfies $\gamma(s) \to \infty$ as $s \to \infty$. For nonlinear functions $\gamma_1$ and $\gamma_2$ defined on $\mathbb{R}_{\geq 0}$, inequality $\gamma_1 \leq \gamma_2$ (or $\gamma_1 < \gamma_2$) represents $\gamma_1(s) \leq \gamma_2(s)$ (or $\gamma_1(s) < \gamma_2(s)$) for all $s > 0$. Id represents identity function.

We employ some notations in graph theory [28] to describe the interconnection of the dynamical network. The interconnection of the dynamical network can be represented with a directed graph by considering the subsystems as *vertices*. We use $\mathcal{V}_i$ to represent the $i$-th vertex (subsystem) in the graph. If the state of the $i$-th subsystem is an input of the $j$-th subsystem, then $\gamma_{ji} \neq 0$ and there is a *directed arc* (interaction) $(\mathcal{V}_i\mathcal{V}_j)$ from $\mathcal{V}_i$ to $\mathcal{V}_j$; otherwise, $\gamma_{ji} = 0$ and there is no arc from $\mathcal{V}_i$ to $\mathcal{V}_j$. A *path* in the directed graph is any sequence of arcs where the final vertex of one is the initial vertex of the next one, denoted by $(\mathcal{V}_i\mathcal{V}_j \cdots \mathcal{V}_k)$. If there exists a path leading from $\mathcal{V}_i$ to $\mathcal{V}_j$, then $\mathcal{V}_j$ is *reachable* from $\mathcal{V}_i$. Specifically, $\mathcal{V}_j$ is reachable from itself. The *reaching set* $RS(\mathcal{V}_j)$ is the set of the vertices from which $\mathcal{V}_j$ is reachable. A closed path with no repeated vertices other than the starting and ending vertices is called a *simple cycle*.

## 3. System formulation

The discrete-time dynamical network studied in this paper is composed of $N$ subsystems in the following form

$$x_i(T+1) = f_i(x(T), u_i(T)), \quad i = 1, \ldots, N; \tag{1}$$

where $x_i \in \mathbb{R}^{n_i}$, $x = [x_1^T, \ldots, x_N^T]^T$, $u_i \in \mathbb{R}^{n_{ui}}$ and $f_i : \mathbb{R}^{\sum_{j=1}^N n_j + n_{ui}} \to \mathbb{R}^{n_i}$ is continuous. $T$ takes values in $\mathbb{Z}_+$. Assume that $f_i(0,0) = 0$, and the external input $u = [u_1^T, \ldots, u_N^T]^T$ is bounded. Denote $f(x,u) = [f_1^T(x,u_1), \ldots, f_N^T(x,u_N)]^T$.

It is assumed that for each $x_i$-subsystem ($i = 1, \ldots, N$), there exists a continuous ISS-Lyapunov function $V_i$ such that the following properties hold.

(1) there exist $\underline{\alpha}_i, \overline{\alpha}_i \in \mathcal{K}_\infty$ such that

$$\underline{\alpha}_i(|x_i|) \leq V_i(x_i) \leq \overline{\alpha}_i(|x_i|), \quad \forall x_i; \tag{2}$$

(2) there exist $\alpha_i \in \mathcal{K}_\infty$, $\sigma_{ij} \in \mathcal{K}_\infty \cup \{0\}$ and $\sigma_{ui} \in \mathcal{K} \cup \{0\}$ such that

$$\begin{aligned} & V_i(f_i(x(T), u_i(T))) - V_i(x_i(T)) \\ & \quad \leq -\alpha_i(V_i(x_i(T))) \\ & \qquad + \max_{j=1,\ldots,N; j\neq i} \left\{\sigma_{ij}\left(V_j\left(x_j(T)\right)\right), \sigma_{ui}(|u_i(T)|)\right\}. \end{aligned} \tag{3}$$

Without loss of generality, we assume that $(\mathrm{Id} - \alpha_i) \in \mathcal{K}$ as in [9]. Define

$$\hat{\gamma}_{ij} = \alpha_i^{-1} \circ (\mathrm{Id} - \rho_i)^{-1} \circ \sigma_{ij} \tag{4}$$

as the ISS gain from $V_j$ to $V_i$, with $\rho_i$ positive definite satisfying $(\mathrm{Id} - \rho_i) \in \mathcal{K}_\infty$.

Correspondingly, we define

$$\hat{\gamma}_{ui} = \alpha_i^{-1} \circ (\mathrm{Id} - \rho_i)^{-1} \circ \sigma_{ui}. \tag{5}$$

**Remark 1.** There are two Lyapunov formulations of discrete-time ISS systems: "gain margin" form and dissipation form (refer to [9,29]). The dissipation form is in the form of (3), while the "gain margin" form can be described as there exist a continuous and positive definite $\alpha_i'$ and $\gamma_{ij}', \gamma_{ui}' \in \mathcal{K} \cup \{0\}$ such that

$$\begin{aligned} V_i(x_i(T)) \geq\ & \max\left\{\gamma_{ij}'\left(V_j\left(x_j(T)\right)\right), \gamma_{ui}'(|u_i(T)|)\right\} \\ & \Rightarrow V_i(f_i(x(T), u_i(T))) - V_i(x_i(T)) \\ & \leq\ -\alpha_i'(V_i(x_i(T))). \end{aligned} \tag{6}$$

However, different from continuous-time systems, the trajectory of the discrete-time system may "jump" out of the region determined by the "gain margin", which means that $\gamma_{ij}'$ and $\gamma_{ui}'$ may not be the real nonlinear gains from $V_j$'s and $u_i$ to $V_i$. However, as done in [12,13], to prove small-gain results the "gain margin" form formulation (6) should be consolidated with

$$\begin{aligned} V_i(x_i(T)) \leq\ & \max\left\{\gamma_{ij}'\left(V_j\left(x_j(T)\right)\right), \gamma_{ui}'(|u_i(T)|)\right\} \\ & \Rightarrow V_i(f_i(x(T), u_i(T))) \\ & \leq\ \left(\mathrm{Id} - \delta_i'\right)\left(\max\left\{\gamma_{ij}'\left(V_j\left(x_j(T)\right)\right),\right.\right. \\ & \qquad \left.\left.\gamma_{ui}'(|u_i(T)|)\right\}\right) \end{aligned} \tag{7}$$

where $\delta_i'$ is continuous and positive definite satisfying $\left(\mathrm{Id} - \delta_i'\right) \in \mathcal{K}_\infty$. □

Combining (6) and (7), the "gain margin" formulation for small-gain analysis is obtained by replacing property (2) with

(2') there exist $\gamma_{ij} \in \mathcal{K}_\infty \cup \{0\}$ and $\gamma_{ui} \in \mathcal{K}_\infty \cup \{0\}$ such that

$$\begin{aligned} & V_i(f_i(x(T), u_i(T))) \\ & \quad \leq (\mathrm{Id} - \delta_i)\left(\max_{j\in\{1,\ldots,N\}\setminus\{i\}} \left\{\begin{array}{l} \gamma_{ij}\left(V_j\left(x_j(T)\right)\right), \\ V_i(x_i(T)), \\ \gamma_{ui}(|u_i(T)|) \end{array}\right\}\right) \end{aligned} \tag{8}$$

where $\delta_i$ is continuous and positive definite satisfying $(\mathrm{Id} - \delta_i) \in \mathcal{K}_\infty$.

For the sake of generality of our main result, we will first consider the dissipation formulation, and then extend the result to the "gain margin" form following a similar idea.