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Basic properties and characterizations of incremental stability prioritizing flow time for a class of hybrid systems

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ABSTRACT

This paper introduces incremental stability notions for a class of hybrid dynamical systems given in terms of differential equations and difference equations with state constraints. The specific class of hybrid systems considered are those that do not have consecutive jumps nor Zeno behavior. The notion of incremental asymptotic stability is used to describe the behavior of the distance between every pair of solutions to the system having stable behavior (incremental stability) and approaching zero asymptotically (incremental attractivity). A version of this notion that is uniform (in hybrid time) with respect to initial conditions is also introduced. These notions prioritize flow time and are illustrated in examples. Basic properties of the class of systems are considered and those implied by the new notions are revealed. An equivalence characterization of the uniform notion is provided in terms of a $\mathcal{KL}$-function. Moreover, sufficient and necessary conditions under which asymptotic stability implies the new incremental notions are provided. We consider the case when the original hybrid system has an asymptotically stable compact set and also the case when an auxiliary hybrid system, which has twice the dimension of the original system, has a diagonal-like set asymptotically stable.

1. Introduction

1.1. Background

Incremental stability is the notion that the distance between any two solutions to the system has stable behavior and converges to zero. Specifically, using a uniform version of such notion,[1] a continuous-time system $\dot{x} = f(x)$ is said to be incrementally stable if there exists a class-$\mathcal{KL}$ function β (a formal definition is provided in Section 1.3) such that every pair of solutions $t \mapsto \phi_1(t)$ and $t \mapsto \phi_2(t)$ to $\dot{x} = f(x)$ satisfies

$$|\phi_1(t) - \phi_2(t)| \le \beta(|\phi_1(0) - \phi_2(0)|, t) \quad (1)$$

for each t in the domain of definition of ϕ_1 and ϕ_2; see, e.g., [1–3]. The bound (1) states that, over their domain of definition, the distance between two solutions is upper bounded by a function of the difference between their initial conditions and also decreases as t gets arbitrarily large (when the domain of definition of the solutions is unbounded to the right). More general forms of this notion in terms of Riemannian distances have been studied in the context of contraction theory; see, e.g., the study of contracting and nonexpansive flows in [4–6], the local arguments in [7], and the regional results in [8] in the context of observer design. Due to often being misinterpreted as a property of convergent systems [9], the authors in [10] provide a rigorous comparison between incremental stability and the property of convergent systems, and conclude that, in general, neither implies the other without extra conditions (uniform convergent systems on positively invariant sets are shown in [10, Theorem 8] to imply incremental stability as defined therein). Of particular interest are the necessary and sufficient conditions in terms of Lyapunov functions reported in [2] for a notion of incremental stability that requires the bound (1) to hold for every pair of solutions.

In recent years, incremental stability has received growing attention due to its use in a wide range of applications and its suitability to study problems in which pairs of trajectories have to converge to each other. Incremental stability-like properties have been used in the study of synchronization [11–13], observer design [8,14], control design [3], symbolic models for nonlinear time-delay systems [15], model reduction [16], as well as the study of convergent systems [17]. Unfortunately, the incremental stability notions and associated results available in the literature do

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[1] Though not made explicitly in the literature, the notion in (1) is uniform with respect to initial conditions by virtue of the properties of $\mathcal{KL}$-functions.

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not apply to systems with variables that can change continuously and, at times, jump discretely (see details in Section 2). These systems, known as *hybrid systems*, are capable of modeling a wide range of complex dynamical systems, including robotic, automotive, and power systems as well as natural processes. The availability of an incremental stability notion for this class of systems would enable the study of similar properties for them as the current notion for continuous-time systems allows. To the best of our knowledge, the notion of incremental stability and its properties for hybrid systems have not been thoroughly studied before, only discussed briefly in [18] for a class of transition systems in the context of bisimulations.

1.2. Contributions

This paper considers the hybrid systems framework presented in [19], where the continuous dynamics (or flows) are modeled using differential equations, while the discrete dynamics (or jumps) are captured by difference equations. Specifically, a hybrid system is denoted by $\mathcal{H}$, has data (C, f, D, g), and is defined by the *hybrid equations* given by

$$\begin{aligned} \dot{z} &= f(z) \quad z \in C, \\ z^+ &= g(z) \quad z \in D, \end{aligned} \tag{2}$$

where $z \in \mathbb{R}^n$ is the state, f defines the flow map capturing the continuous dynamics, and C defines the flow set on which f is effective. The map g defines the jump map and models the discrete behavior, while D defines the jump set, which is the set of points from where jumps are allowed. While set-valued flow and jump maps could certainly be considered in (2), which would lead to *hybrid inclusions*, there is no loss of generality in considering single-valued maps since, as it will be shown in Section 3, uniqueness of solutions (of their x component) is a necessary condition for incremental stability; see Lemma 3.2. It should be noted that a robust stability theory for hybrid systems modeled as hybrid inclusions was developed in [19].

Motivated by the broad application of incremental stability and the recent advancements in the theory of hybrid systems given as in (2), incremental stability notions are introduced, and many of their basic properties, characterizations, and equivalences are presented. As will be discussed in Section 2, the jumps imposed by the difference equations make it inherently difficult to define incremental stability for hybrid systems. We address this issue by prioritizing flow time in the definition of incremental stability. More precisely, for a class of hybrid systems, the contributions of this paper include the following:

(1) Notions of incremental stability: Non-uniform and uniform incremental (global) asymptotic stability notions with respect to full or partial state in terms of stability and attractivity are given.
(2) Equivalent characterization of incremental stability: An equivalent characterization of incremental uniform (global) asymptotic stability in terms of a $\mathcal{KL}$ bound on the distance between every two maximal solutions[2] is provided.
(3) Equivalent auxiliary system for the study of incremental stability through asymptotic stability of the diagonal-like set: An auxiliary hybrid system with twice the dimension of the original system is defined and the asymptotic stability properties of the diagonal-like set for the auxiliary system are related to incremental stability of the original system.
(4) Relationships between incremental asymptotic stability and asymptotic stability properties: Links between properties involved in the definition of incremental asymptotic stability and of asymptotic stability of a system are revealed. In particular, it is shown that uniform global attractivity of a singleton set implies incremental uniform attractivity, and that strong forward invariance of a compact set plus uniform incremental stability of a system leads to uniform global asymptotic stability of the said set for a hybrid system with regular enough data.

This article contains new incremental stability results for the hybrid framework in [19]. A preliminary version of these results is shown in [20]. We are not aware of any other previous results on incremental stability for hybrid systems (other than the discussions in [18] we mentioned in Section 1.1).

1.3. Organization and notation

The remainder of this paper is organized as follows. In Section 2, the incremental stability notions are introduced for a class of hybrid systems. The auxiliary hybrid system is also introduced in Section 2. In Section 3, the main results are presented, namely, items (2)–(4) listed in Section 1.2.

The notation used throughout the paper is as follows. Given a set $S \subset \mathbb{R}^n$, the closure of S is denoted by $\overline{S}$, and the interior of S is denoted by $\operatorname{int} S$. The set of nonnegative real and natural numbers are $\mathbb{R}_{\geq 0} := [0, \infty)$ and $\mathbb{N} := \{0, 1, \ldots\}$, respectively. Given vectors $\nu \in \mathbb{R}^n$, $w \in \mathbb{R}^m$, $|\nu|$ defines the Euclidean vector norm $|\nu| = \sqrt{\nu^\top \nu}$, and $[\nu^\top \; w^\top]^\top$ is equivalent to (ν, w). Given a function $f : \mathbb{R}^m \to \mathbb{R}^n$, its domain of definition is denoted by $\operatorname{dom} f$, i.e., $\operatorname{dom} f := \{x \in \mathbb{R}^m : f(x) \text{ is defined}\}$. The range of f is denoted by $\operatorname{rge} f$, i.e., $\operatorname{rge} f := \{f(x) : x \in \operatorname{dom} f\}$. The right limit of the function f is defined as $f^+(x) := \lim_{\nu \to 0^+} f(x + \nu)$ if it exists. Given a point $y \in \mathbb{R}^n$ and a closed set $\mathcal{A} \subset \mathbb{R}^n$, $|y|_{\mathcal{A}} := \inf_{x \in \mathcal{A}} |x - y|$. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a class-$\mathcal{K}_\infty$ function, also written $\alpha \in \mathcal{K}_\infty$, if α is zero at zero, continuous, strictly increasing, and unbounded. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a class-$\mathcal{KL}$ function, also written $\beta \in \mathcal{KL}$, if it is nondecreasing in its first argument, nonincreasing in its second argument, $\lim_{r \to 0^+} \beta(r, s) = 0$ for each $s \in \mathbb{R}_{\geq 0}$, and $\lim_{s \to \infty} \beta(r, s) = 0$ for each $r \in \mathbb{R}_{\geq 0}$.

2. Incremental stability notions

In this paper, for hybrid systems $\mathcal{H}$ as in (2), we are interested in characterizing the incremental stability property, namely, the notion that the distance between every pair of maximal solutions to the system has stable behavior and approaches zero asymptotically. To highlight the intricacies of this property in the hybrid setting, consider the so-called bouncing ball system. This is a canonical example of hybrid systems to which every maximal solution is Zeno[3] and converges to the origin; see [19, Example 1.1 and 2.12] for more details. Consider two solutions to this system, given by ϕ_1 and ϕ_2, from initial conditions $\phi_1(0, 0) = (1, 0)$ (ball initialized close to the ground with zero initial velocity) and

[2] A solution ϕ to $\mathcal{H}$ is parametrized by $(t, j) \in \mathbb{R}_{\geq 0} \times \mathbb{N}$, where t denotes ordinary time and j denotes jump time. The domain of a solution $\operatorname{dom} \phi \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a hybrid time domain if for every $(T, J) \in \operatorname{dom} \phi$, the set $\operatorname{dom} \phi \cap ([0, T] \times \{0, 1, \ldots, J\})$ can be written as the union of sets $\cup_{j=0}^{J} (I_j \times \{j\})$, where $I_j := [t_j, t_{j+1}]$ for some time sequence $0 = t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_{J+1}$. The t_j's with $j > 0$ define the time instants when the state of the hybrid system jumps and j counts the number of jumps. A solution to $\mathcal{H}$ is called maximal if it cannot be extended, i.e., it is not a truncated version of another solution. It is called complete if its domain is unbounded.

[3] A solution is Zeno if it is complete and its domain is bounded in the t direction. See [19] for more details.