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On the relation between strict dissipativity and turnpike properties[☆]

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ABSTRACT

For discrete time nonlinear systems we study the relation between strict dissipativity and so called turnpike-like behavior in optimal control. Under appropriate controllability assumptions we provide several equivalence statements involving these two properties. The relation of strict dissipativity to an exponential variant of the turnpike property is also studied.

1. Introduction

Dissipativity and strict dissipativity have been recognized as important systems theoretic properties since their introduction by Willems in [1,2]. Dissipativity formalizes the fact that a system cannot store more energy than supplied from the outside, strict dissipativity in addition requires that a certain amount of the stored energy is dissipated to the environment. As such, dissipativity like properties are naturally linked to stability considerations and thus particular forms of dissipativity like, e.g., passivity naturally serve as tools for the design of stabilizing controllers [3,4]. In recent years, dissipativity properties turned out to be an important ingredient for understanding the stability behavior of economic model predictive control (MPC) schemes, [5–8]. Loosely speaking, they allow for the construction of a Lyapunov function from an optimal value function also in case the stage cost of the optimal control problem under consideration is not positive definite. Moreover, they are intimately related to the existence of steady states at which the system is optimally operated, see [9–11]. The present paper is similar to the last reference in the sense that necessary and sufficient conditions for strict dissipativity are derived in terms of properties of certain optimal control problems. However, in contrast to [11] in which optimal operation at steady states is considered, in this paper we focus on the so called turnpike property and more general turnpike-like behavior.

The turnpike property has been observed and studied already in the 1940s and 1950s by von Neumann [12] and by Dorfman, Samuelson and Solow [13] in the context of economic optimal control problems. It formalizes the phenomenon that optimally controlled trajectories "most of the time" stay close to an optimal steady state. In this paper, we use variants of this property which also demand that trajectories which are nearly optimal or whose value lies near the steady state value exhibit this behavior (see Definition 2.2, for details). Given its usefulness, e.g., in the design of optimal trajectories [14] or – again – in the analysis of economic MPC schemes [15,7,8], both in discrete and continuous time, it is of no surprise that there is a rich body of literature on conditions which ensure that the turnpike property does indeed occur, see, e.g., the monographs [16,17] or the recent papers [18,19] and the references therein.

Although the deep relation between dissipativity and optimal control was studied already in the early days of dissipativity theory [20], it seems that only in [7, Theorems 5.3 and 5.6] it was observed that strict dissipativity plus a suitable controllability property is sufficient for the occurrence of turnpike-like behavior (though there are earlier similar results, like [16, Theorem 4.2], observing that Assumption 4.2 in this reference is essentially a linearized version of strict dissipativity). Likewise, it is easily seen that strict dissipativity implies that the system is optimally operated at a steady state. Motivated by recently developed converse statements, i.e., results which show that optimal operation at a steady state may also imply dissipativity [9–11], in this paper for general nonlinear discrete time systems we investigate whether the implication "strict dissipativity ⇒ turnpike-like behavior" also admits for converse statements. Under suitable controllability assumptions we show that this is indeed the case and we provide two

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main theorems which provide equivalence relations between strict dissipativity and turnpike-like behavior under different structural assumptions. Moreover, we show that the exponential turnpike property [18] also implies strict dissipativity.

The paper is organized as follows. Section 2 defines the problem setting and gives precise mathematical definitions for the various properties used in this paper. Section 3 summarizes results from the literature and provides auxiliary technical results. The main theorems and their proofs are given in Section 4. Section 5 concludes the paper.

2. Setting and definitions

We consider discrete time nonlinear systems of the form

$$x(k+1) = f(x(k), u(k)), \quad x(0) = x_0 \tag{2.1}$$

for a continuous map $f : X \times U \to X$, where X and U are normed spaces. We impose the constraints $(x, u) \in \mathbb{Y} \subseteq X \times U$ on the state x and the input u and define $\mathbb{X} := \{x \in X \mid \exists u \in U : (x, u) \in \mathbb{Y}\}$ and $\mathbb{U} := \{u \in U \mid \exists x \in X : (x, u) \in \mathbb{Y}\}$. A control sequence $u \in \mathbb{U}^N$ is called admissible for $x_0 \in \mathbb{X}$ if $(x(k), u(k)) \in \mathbb{Y}$ for $k = 0, \ldots, N-1$ and $x(N) \in \mathbb{X}$. In this case, the corresponding trajectory $x(k)$ is also called admissible. The set of admissible control sequences is denoted by $\mathbb{U}^N(x_0)$. Likewise, we define $\mathbb{U}^\infty(x_0)$ as the set of all control sequences $u \in \mathbb{U}^\infty$ with $(x(k), u(k)) \in \mathbb{Y}$ for all $k \in \mathbb{N}_0$. In order to keep the presentation technically simple, we assume that $\mathbb{X}$ is controlled invariant, i.e., that $\mathbb{U}^\infty(x_0) \neq \emptyset$ for all $x_0 \in \mathbb{X}$. We expect that our results remain true if one restricts the initial values under consideration to the viability kernel $\mathbb{X}_\infty := \{x_0 \in \mathbb{X} \mid \mathbb{U}^\infty(x_0) \neq \emptyset\}$, however, the technical details of this extension are beyond the scope of this paper. The trajectories of (2.1) are denoted by $x_u(k, x_0)$ or simply by $x(k)$ if there is no ambiguity about x_0 and u.

Given a continuous stage cost $\ell : \mathbb{Y} \to \mathbb{R}$ and a time horizon $K \in \mathbb{N}$, we consider the optimal control problem

$$\min_{u \in \mathbb{U}^K(x_0)} J_K(x_0, u) \quad \text{with } J_K(x_0, u) = \sum_{k=0}^{K-1} \ell(x(k), u(k)) \tag{2.2}$$

subject to (2.1). By $V_K(x_0) := \inf_{u \in \mathbb{U}^K(x_0)} J_K(x_0, u)$ we denote the optimal value function of the problem. For Definition 2.2(c) and (d), we will need the existence of the global minimum in (2.2). However, for most of the statements in this paper its existence is not needed. Moreover, in those statements which require the existence of a minimizing control sequence we do not need its uniqueness.

The next definition formalizes the strict dissipativity property, originally introduced by Willems [1] in continuous time and by Byrnes and Lin [21] in the discrete time setting of this paper. While one may formulate dissipativity with respect to arbitrary supply rates $s : X \times U \to \mathbb{R}$, here we restrict ourselves to supply rates of the form $s(x, u) = \ell(x, u) - \ell(x^e, u^e)$ for ℓ from (2.2) and a steady state (x^e, u^e) of (2.1), which will be the form used throughout this paper. We recall that $(x^e, u^e) \in \mathbb{Y}$ is a steady state of (2.1) if $f(x^e, u^e) = x^e$.

Definition 2.1. Given a steady state (x^e, u^e), the optimal control problem (2.1), (2.2) is called *strictly dissipative* with respect to the supply rate $\ell(x, u) - \ell(x^e, u^e)$ if there exists a storage function $\lambda : \mathbb{X} \to \mathbb{R}$ bounded from below and a function $\rho \in \mathcal{K}_\infty$ such that

$$\ell(x, u) - \ell(x^e, u^e) + \lambda(x) - \lambda(f(x, u)) \geq \rho(\|x - x^e\|) \tag{2.3}$$

holds for all $(x, u) \in \mathbb{Y}$ with $f(x, u) \in \mathbb{X}$. The system is called *dissipative* if the same property holds with $\rho \equiv 0$.

The next definition formalizes four variants of turnpike-like behavior. The behavior of the trajectories described in the four definitions is essentially identical and in all cases demands that the trajectory stays in a neighborhood of a steady state most of the time. What distinguishes the definitions are the conditions on the trajectories under which we demand this property to hold and in case of (d) the bound on the size of the neighborhood.

Definition 2.2. Consider the optimal control problem (2.1), (2.2) and let (x^e, u^e) be a steady state of (2.1).

(a) The optimal control problem is said to have *turnpike-like behavior of near steady state solutions*, if there exist $C_a > 0$ and $\rho \in \mathcal{K}_\infty$ such that for each $x \in \mathbb{X}$, $\delta > 0$ and $K \in \mathbb{N}$, each control sequence $u \in \mathbb{U}^K(x)$ satisfying $J_K(x, u) \leq K\ell(x^e, u^e) + \delta$ and each $\varepsilon > 0$ the value $Q_\varepsilon := \#\{k \in \{0, \ldots, K-1\} \mid \|x_u(k, x) - x^e\| \leq \varepsilon\}$ satisfies the inequality $Q_\varepsilon \geq K - (\delta + C_a)/\rho(\varepsilon)$.

(b) The optimal control problem is said to have the *turnpike-like behavior of near optimal solutions*, if there exist $C_d > 0$ and $\rho \in \mathcal{K}_\infty$ such that for each $x \in \mathbb{X}$, $\delta > 0$ and $K \in \mathbb{N}$, each control sequence $u \in \mathbb{U}^K(x)$ satisfying $J_K(x, u) \leq V_K(x) + \delta$ and each $\varepsilon > 0$ the value $Q_\varepsilon := \#\{k \in \{0, \ldots, K-1\} \mid \|x_u(k, x) - x^e\| \leq \varepsilon\}$ satisfies the inequality $Q_\varepsilon \geq K - (\delta + C_d)/\rho(\varepsilon)$.

(c) The optimal control problem is said to have the *(steady state) turnpike property*, if there exist $C_b > 0$ and $\rho \in \mathcal{K}_\infty$ such that for each $x \in \mathbb{X}$ and $K \in \mathbb{N}$ and any corresponding optimal control sequence $u^\star \in \mathbb{U}^K(x)$ and $\varepsilon > 0$ the value $Q_\varepsilon := \#\{k \in \{0, \ldots, K-1\} \mid \|x_{u^\star}(k, x) - x^e\| \leq \varepsilon\}$ satisfies the inequality $Q_\varepsilon \geq K - C_b/\rho(\varepsilon)$.

(d) The optimal control problem is said to have the *exponential input-state turnpike property* if there is $C_c > 0$ and $\eta \in (0, 1)$ such that for each $x \in \mathbb{X}$ and $K \in \mathbb{N}$ and any corresponding optimal control sequence $u^\star \in \mathbb{U}^K(x)$ the inequality $\max\{\|x_{u^\star}(k, x) - x^e\|, \|u^\star(k) - u^e\|\} \leq C_c \max\{\eta^k, \eta^{K-k}\}$ holds for all but at most C_c times $k \in \{0, \ldots, K-1\}$.

The turnpike-like behavior of near steady state solutions (a) ensures that each trajectory for which the associated cost is close to the steady state value stays most of the time in a neighborhood of x^e. However, it does not demand that such trajectories exist for initial values $x \neq x^e$. The turnpike-like behavior of near optimal solutions (b) requires the same property to hold for all trajectories whose associated cost is close to the optimal one, while the (steady-state) turnpike property (c) demands this behavior only for the optimal trajectories. The exponential input-state turnpike property (d) strengthens this property in two ways: the imposed inequality involves x and u and the distance from the steady state is required to decrease exponentially fast. While (c) is the property that is most often found in the literature when turnpike properties are discussed, it turns out that for the purpose of this paper the other three properties are more suitable.

It is straightforward to see that (d) implies (c) and that (b) implies (c) with $C_b = C_d$. Moreover, if there exists a constant $D > 0$ with $V_K(x) \leq K\ell(x^e, u^e) + D$ for all $x \in \mathbb{X}$ then (a) implies (b) with $C_d = C_a + D$, cf. Lemma 3.9. This property and its converse variant are formalized as follows.

Definition 2.3. Consider the optimal control problem (2.1), (2.2) and let (x^e, u^e) be a steady state of (2.1).

(a) We say that x^e is *cheaply reachable* if there exists a constant $D > 0$ with $V_K(x) \leq K\ell(x^e, u^e) + D$ for all $x \in \mathbb{X}$ and all $K \in \mathbb{N}$.

(b) We say that the system is *non-averaged steady state optimal at* (x^e, u^e) if there exists a constant $E > 0$ with $V_K(x) \geq K\ell(x^e, u^e) - E$ for all $x \in \mathbb{X}$ and all $K \in \mathbb{N}$.