Systems & Control Letters 90 (2016) 54–60

Contents lists available at ScienceDirect

Systems & Control Letters

journal homepage: www.elsevier.com/locate/sysconle

Linear switching systems with slow growth of trajectories☆

Vladimir Yu. Protasov

Department of Mechanics and Mathematics, Moscow State University and Faculty of Computer Science of National Research University Higher School of Economics, Russian Federation

ARTICLE INFO

Article history:
Received 24 July 2015
Received in revised form
20 January 2016
Accepted 21 January 2016
Available online 22 February 2016

Keywords:
Dynamical systems
Linear switching systems
Growth of trajectories
Stability
Resonance

ABSTRACT

We prove the existence of positive linear switching systems (continuous time), whose trajectories grow to infinity, but slower than a given increasing function. This implies that, unlike the situation with linear ODE, the maximal growth of trajectories of linear systems may be arbitrarily slow. For systems generated by a finite set of matrices, this phenomenon is proved to be impossible in dimension 2, while in all bigger dimensions the sublinear growth may occur. The corresponding examples are provided and several open problems are formulated.

1. Introduction

For a linear ordinary differential equation (ODE) with constant coefficients $\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t),\ t \in \mathbb{R}_+$, where $\boldsymbol{x}(t) \in \mathbb{R}^d$ and A is a real $d \times d$ matrix, the fastest growth of trajectories is exponential with the parameter $\sigma = \sigma(A)$ equal to the spectral abscissa, i.e., the biggest real part of eigenvalues of A. In particular, the system is stable, i.e., all its trajectories converge to zero as $t \to \infty$, precisely when $\sigma < 0$. In case $\sigma = 0$, the system is bounded, i.e., has bounded trajectories, apart from the case of resonance, when there are nontrivial Jordan blocks of eigenvalues with zero real part. In that case, the fastest growth is always polynomial with integer degree: $\|\boldsymbol{x}(t)\| \asymp t^{\ell-1},\ t \to \infty$, where ℓ is the largest size of those Jordan blocks called the *resonance degree* of the system. In particular, *every system is either bounded or has at least linear growth*.

The same situation occurs for discrete systems $\boldsymbol{x}(t+1) = A\boldsymbol{x}(t),\ t \in \mathbb{N} \cup \{0\}$. If the spectral radius $\rho(A)$ is equal to one, then the trajectories are unbounded if and only if there are nontrivial Jordan blocks corresponding to the largest by modulus eigenvalues. The fastest growth is again $t^{\ell-1}$, where ℓ is the largest size of those blocks.

The resonance phenomenon have countless applications. Its analysis becomes much more difficult, when the matrix A may depend on time and take values from a given compact *control set* of matrices $\mathcal{A}$. In this case, we obtain a dynamical system of the form

$$\begin{cases} \dot{\boldsymbol{x}}(t) = A(t)\boldsymbol{x}(t); \\ \boldsymbol{x}(0) = \boldsymbol{x}_0, \end{cases} \tag{1}$$

where $A(\cdot) : [0, +\infty) \to \mathcal{A}$, is a measurable function called the *switching law*. This is a continuous *linear switching system* (LSS). A solution $\boldsymbol{x} : \mathbb{R}_+ \to \mathbb{R}^d$ of this system is called its *trajectory* corresponding to that switching law and to the initial condition $\boldsymbol{x}(0) = \boldsymbol{x}_0$. For a single-matrix set $\mathcal{A} = \{A\}$, the LSS becomes a usual linear ODE. There is an extensive bibliography on the theory of LSS and many applications in control, dynamical systems, engineering, economics, biology, etc., see [1–7] and references therein.

The system is *stable* if $\boldsymbol{x}(t) \to 0$ as $t \to \infty$ for every switching law $A(\cdot)$. Thus, the stability only depends on the compact family $\mathcal{A}$. Denote $F(t) = F_{\mathcal{A}}(t) = \sup\{\|\boldsymbol{x}(t)\| \mid \|\boldsymbol{x}_0\| = 1\}$, where the supremum is computed over all switching laws $A(\cdot)$. The system is stable precisely when $F(t) \to 0$ as $t \to \infty$. The *Lyapunov exponent* $\sigma(\mathcal{A})$ is defined as

$$\sigma(\mathcal{A}) = \limsup_{t\to\infty} \frac{1}{t} \ln F(t).$$

For a single-matrix set $\mathcal{A}$, this becomes the spectral abscissa. The system is stable if and only if $\sigma(\mathcal{A}) < 0$ [6]. If $\sigma(\mathcal{A}) > 0$, then there are unbounded trajectories with an exponential growth as $t \to \infty$. In the boundary case $\sigma(\mathcal{A}) = 0$, the system is never stable, i.e., there is at least one trajectory that does not converge to the origin as $t \to \infty$ [1]. We focus on the question whether the system is *bounded* in this case, i.e., all its trajectories are bounded.

☆ The work is supported by RFBR Grant Nos. 14-01-00332 and 16-04-00832, and by the grant of Dynasty foundation.

E-mail address: v-protassov@yandex.ru.

http://dx.doi.org/10.1016/j.sysconle.2016.01.006

Definition 1. In the case $\sigma(\mathcal{A}) = 0$, system (1) is called marginally stable if it is bounded, otherwise it is called marginally unstable.

It is known that the trajectories of marginally unstable systems can grow at most polynomially, and, moreover, $\|x(t)\| \le C(1 + t^{d-1}),\ t \in \mathbb{R}_+$, where d is the dimension. Similarly to the case of one matrix, a generic system with $\sigma(\mathcal{A}) = 0$ is bounded. It can be unbounded only if it is reducible, i.e., all matrices from $\mathcal{A}$, in some common basis, have the block upper triangular form:

$$A = \begin{pmatrix} A^{(1)} & * & \cdots & * \\ 0 & A^{(2)} & * & \vdots \\ \vdots & & \ddots & * \\ 0 & \cdots & 0 & A^{(n)} \end{pmatrix}, \qquad (2)$$

with irreducible families $\mathcal{A}^{(i)} = \{A^{(i)},\ A \in \mathcal{A}\}$ in all the diagonal blocks, $i = 1, \ldots, n$. It is known that $\sigma(\mathcal{A}) = \max_{i=1,\ldots,n} \sigma(\mathcal{A}^{(i)})$ [1]. The number r of blocks for which this maximum is attained is called the *valency* of the system. It was shown in [3] that $F(t) \le C(1 + t^{r-1})$.

Similarly to (1), a *discrete time switching system* is the following difference equation:

$$\begin{cases} \boldsymbol{x}(t+1) = A(t)\,\boldsymbol{x}(t), & t \in \mathbb{N} \cup \{0\}; \\ \boldsymbol{x}(0) = \boldsymbol{x}_0, \end{cases} \qquad (3)$$

where the switching law $A(t)$ is a sequence of elements from $\mathcal{A}$. The notions of trajectory, stability, boundedness, the growth $F(t)$, etc., are directly extended to discrete systems. A similar estimate of growth $F(t) \le C(1 + t^{r-1})$ takes place for discrete systems [8].

1.1. Our results

There is a lot of similarity between linear ODE and general linear switching systems. In case $\sigma = 0$, the system is "typically" bounded. The existence of unbounded trajectories requires reducibility of all matrices to the form (2) and coincidence of Lyapunov exponents of several blocks (an analogue of resonance). The growth of trajectories in this case is at most polynomial with degree bounded above by the total number of blocks with the largest Lyapunov exponent (an analogue of the resonance degree). See [3] for sharpening those results, which revealed even more similarity with the single-matrix case. Our main problem is whether the growth is *at least* polynomial? We focus on continuous LSS and address the following questions:

Is the maximal growth of trajectories of a marginally unstable LSS always polynomial with integer degree? In particular, *whether an unbounded system has at least linear growth*, as in the single-matrix case? *Can the function $F(t)$ grow slower than polynomially, say, logarithmically?*

We shall see that the answers depend on whether the family $\mathcal{A}$ is finite (i.e., consists of finitely many matrices) or infinite. The finite case turns out to be more difficult and interesting. Note that the system with a control set $\mathcal{A}$ has the same growth of trajectories as that with $\mathrm{co}(\mathcal{A})$, where $\mathrm{co}(\cdot)$ is the convex hull [6]. Hence, the case of finite $\mathcal{A}$ is essentially the same as the case of a polytope set $\mathcal{A}$.

In Theorem 2, we show the existence of systems with arbitrarily slow growth. For every positive increasing to infinity function $f(t)$, there is an unbounded LSS for which $F(t)$ grows slower than $f(t)$ as $t \to \infty$. Such systems exist in all dimensions $d \ge 2$ and can be constructed positive (i.e., all their matrices are Metzler). However, in case $d = 2$, this phenomenon never occurs for finite systems. By Theorem 1 proved in Section 2, if the set of 2×2 matrices $\mathcal{A}$ is finite (polytope) or infinite but not containing zero, then a marginally unstable system always has linear growth: $F(t) \asymp t$ as $t \to \infty$. Already those results demonstrate a crucial difference between finite (polytope) and general cases. In view of Theorems 1 and 2, the slow growth phenomenon emerges because of matrices of small norms in $\mathcal{A}$. A question arises if the slow growth possible for finite families in higher dimensions? Theorem 3 gives an affirmative answer. It provides an example of two matrices that generate an LSS in $\mathbb{R}^3$ with the maximal growth close to $\sqrt{t}$. Thus, starting with the dimension 3, a sublinear growth of $F(t)$ is possible even for finite families. The proof of Theorem 3 is surprisingly difficult and required some special technique (Section 5).

Theorems 1–3 answer two open questions formulated in [9]. Their possible generalizations (for a slower growth or for positive systems) are left as open problems in Section 6. To prove Theorem 1 we derive a criterion of marginal stability for two-dimensional finite systems (Proposition 1). An open problem is formulated in Section 6 on extensions of that result to higher dimensions.

1.2. Related works and known results

Resonance and marginal instability of linear switching systems have been analyzed in the literature in various contexts. In the study of wavelets, refinement functional equations, and affine fractal curves, marginal stability is responsible for Lipschitz continuity and for boundedness of variation of solutions [10,11,8]. It is important for trackability of autonomous agents in sensor networks [12], in classifications of finite semigroups of integer matrices [13], in the problem of asymptotic growth of some regular sequences [14], in the stability analysis of LSS [3,7,9], etc.

The maximal rate of growth of marginally unstable systems were estimated in succession in several works (see [3] for the discussion and references). Those results give only the upper bounds of the polynomial growth and necessary conditions for marginal instability. Criteria of marginal instability are known only in some favorable cases [3,9,7]. For discrete systems, possible rates of growth of trajectories were found in several special cases. It was proved to be polynomial with integer degree in case of integer nonnegative matrices [13], and then extended to all integer matrices [14]. The first examples of discrete systems with sublinear growth were presented in [15] for general LSS and in [9] for finite ones. For continuous LSS those constructions are not applicable and the answer was unknown [9, open problem 3].

1.3. Notation

In the sequel we consider only continuous LSS. We identify a system with its control set $\mathcal{A}$ generating it. A matrix is called Metzler if all its off-diagonal elements are nonnegative. A system is positive if all matrices from $\mathcal{A}$ are Metzler. See [4,5] for results on positive systems.

We use the standard notation $g(t) = o(f(t))$ and $g(t) = O(f(t))$ as $t \to \infty$ meaning $\lim_{t\to\infty}\left|\frac{g(t)}{f(t)}\right| = 0$ and $\limsup_{t\to\infty}\left|\frac{g(t)}{f(t)}\right| < \infty$ respectively. We say that a system grows slower (not faster) than a positive function $f(t)$ if $F(t) = o(f(t))$ (respectively, $F(t) = O(f(t))$) as $t \to \infty$. Two values are asymptotically equivalent ($f(t) \asymp F(t)$) if they grow not faster each other. We use bold letters to denote vectors, $\|\cdot\|$ is a Euclidean norm.

2. Two-dimensional finite systems: the marginal instability means linear growth

We begin our analysis with two-dimensional LSS. In this case, any unbounded system has at least linear growth, provided the family $\mathcal{A}$ does not contain a sequence that tends to zero.