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Stochastic minimum-energy control

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ABSTRACT

We give the solution to the minimum-energy control problem for linear stochastic systems. The problem is as follows: given an exactly controllable system, find the control process with the minimum expected energy that transfers the system from a given initial state to a desired final state. The solution is found in terms of a certain forward–backward stochastic differential equation of Hamiltonian type.

1. Introduction

The notion of *controllability* was introduced by Kalman [1] and it characterizes the ability of controls to transfer a system from a given initial state to a desired final state. When the system is completely controllable, there are many controls that can achieve such a transfer of the system state. This naturally leads to the problem of choosing the "best" control that performs this task. Another contribution of Kalman [1] was the solution of this problem for linear deterministic systems and using the quadratic cost as an optimality criterion. These kinds of problems are known as the *minimum-energy control* problems (see also [2–5]).

There has been a great progress in extending Kalman's results from the deterministic to the stochastic settings. Various different notions of controllability for stochastic systems have been introduced (see, for example, [6–12]), and the stochastic linear-quadratic (LQ) regulator continues to be developed in new stochastic settings (see, for example, [13–20]). On the other hand, much less progress has been made on the *stochastic* minimum-energy control problem. This problem is particularly difficult in the stochastic setting since the terminal value is a random variable rather than a fixed number. The papers by Klamka [9,21–23] consider this problem for linear stochastic systems with *additive* noise only, and thus do not cover the important class of stochastic systems with *multiplicative* noise, which appear in many applications. Examples of stochastic systems with multiplicative noise can be found in mathematical finance, where the self-financing portfolio is such a stochastic control system (see, e.g., [24–27]), in mechanical systems subject to random parameter variation (see, e. g., [12]), or in altitude control problems (see, e.g., [28]).

In this paper we formulate and solve the minimum-energy control problem for linear stochastic systems with *multiplicative* noise. The notion of controllability that we adapt is that of *exact controllability*, as introduced by Peng [8] (see also [10–12, 29,30]). This notion of controllability is a faithful extension of Kalman's notion of complete controllability to stochastic systems. The difference between these two definitions is that in the case of exact controllability the terminal state can be a random variable rather than a fixed number.

The precise formulation of the stochastic minimum-energy control problem is given in the next section. This is followed by the proof of solvability for a Hamiltonian system and its relation with exact controllability. Section 4 contains the solution to the stochastic minimum energy control problem, which is illustrated with a couple of examples. As an extension of this result, we give the solution to the stochastic LQ regulator problem with a fixed final state in the final section.

2. Problem formulation

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t,\ t \geq 0), \mathbb{P})$ be a given complete filtered probability space on which the scalar standard Brownian motion $(W(t),\ t \geq 0)$ is defined. We assume that $\mathcal{F}_t$ is the augmentation of $\sigma\{W(s) : 0 \leq s \leq t\}$ by all the $\mathbb{P}$-null sets of $\mathcal{F}$. If $\xi : \Omega \to \mathbb{R}^n$ is an $\mathcal{F}_T$-measurable random variable such that $\mathbb{E}[|\xi|^2] < \infty$, we write $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}^n)$. If $f : [0, T] \times \Omega \to \mathbb{R}^n$ is an $\{\mathcal{F}_t\}_{t\geq 0}$ adapted process and if $\mathbb{E}\int_0^T |f(t)|^2 dt < \infty$, we write $f(\cdot) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^n)$; if $f(\cdot)$ has a.s. continuous sample paths and $\mathbb{E}\sup_{t\in[0,T]} |f(t)|^2 < \infty$, we write $f(\cdot) \in L^2_{\mathcal{F}}(\Omega; C(0, T; \mathbb{R}^n))$; if

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$f(\cdot)$ is uniformly bounded (i.e. $\mathrm{esssup}_{t\in[0,T]}|f(t)| < \infty$), we write $f(\cdot) \in L^\infty(0,T;\mathbb{R}^n)$.

Consider the linear stochastic control system:

$$\begin{cases} dx(t) = [A(t)x(t) + B(t)u(t)]dt \\ \qquad + [C(t)x(t) + D(t)u(t)]dW(t) \\ x(0) = x_0 \in \mathbb{R}^n, \quad \text{is given.} \end{cases} \tag{2.1}$$

We assume that $A(\cdot), C(\cdot) \in L^\infty(0,T;\mathbb{R}^{n\times n})$, and $B(\cdot), D(\cdot) \in L^\infty(0,T;\mathbb{R}^{n\times m})$. If the control process $u(\cdot)$ belongs to $L^2_{\mathcal{F}}(0,T;\mathbb{R}^m)$, then (2.1) has a unique strong solution $x(\cdot) \in L^2_{\mathcal{F}}(\Omega; C(0,T;\mathbb{R}^n))$ (see, e.g. Theorem 1.6.14 of [19]).

For a given $\xi \in L^2(\Omega, \mathcal{F}_T, P; \mathbb{R}^n)$, we are interested in the following subset of control processes:

$$\mathcal{U}_\xi \equiv \left\{u(\cdot) \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^m) : x(T) = \xi \text{ a.s.}\right\}.$$

Minimum-energy control problem. *Let $R(\cdot) \in L^\infty(0,T;\mathbb{R}^{m\times m})$ be a given symmetric matrix such that $R(t) > 0$, a.e. $t \in [0,T]$. For any given $x_0 \in \mathbb{R}^n$ and $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}^n)$ find the control process $u(\cdot) \in \mathcal{U}_\xi$ that minimizes the cost functional*

$$J(u(\cdot)) = \mathbb{E}\int_0^T u'(t)R(t)u(t)dt. \tag{2.2}$$

This is clearly the stochastic version of the Kalman's minimum energy control problem. A related problem was considered by Klamka [9,21–23]. However, Klamka considers linear stochastic systems with additive noise only, whereas (2.1) has a multiplicative noise. Our approach to solving the stochastic minimum-energy control problem is different from the operator-theoretic method of Klamka, and is based on a forward–backward stochastic differential equation of a Hamiltonian type.

In order to ensure that the set $\mathcal{U}_\xi$ is not empty, we make some assumptions on the controllability of (2.1). Out of the many possible notions of controllability for stochastic systems, we employ the notion of *exact controllability* as introduced by Peng [8].

Definition 1. System (2.1) is called exactly controllable at time $T > 0$ if for any $x_0 \in \mathbb{R}^n$ and $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}^n)$, there exists at least one control $u(\cdot) \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^m)$, such that the corresponding trajectory $x(\cdot)$ satisfies the initial condition $x(0) = x_0$ and the terminal condition $x(T) = \xi$, a.s.

We solve the minimum-energy control problem under the following two assumptions.

(A1) The system (2.1) is exactly controllable at time $T > 0$.
(A2) There exists an *invertible* matrix $M(\cdot) \in L^\infty(0,T;\mathbb{R}^{m\times m})$ such that $D(t)M(t) = [I, 0]$.

Assumption A1 ensures that the set $\mathcal{U}_\xi$ is not empty. An example of a stochastic control system that satisfies this assumption (as well as assumption A2) is the self-financing portfolio (see, e.g., [24,25]) in a market with one riskless and one risky asset, with equation

$$\begin{cases} dy(t) = [ry(t) + bu(t) - c(t)]dt + \sigma u(t)dW(t) \\ y(0) = y_0 \in \mathbb{R}, \quad \text{is given,} \end{cases}$$

where $y(t)$ is the portfolio value (investor's wealth), and the controls $c(t)$ and $u(t)$ represent the consumption rate and the wealth invested in the risky asset, respectively. One can easily check that Peng's [8] necessary and sufficient condition for exact controllability is satisfied in this case.

Assumption A2 implies that $m \geq n$, i.e. the number of control inputs to the system is at least as large as the number of the states of the system. This may appear as a strong assumption when compared with the minimum-energy control problem of deterministic systems. However, at least when the matrix $D(\cdot)$ has continuous coefficients, this assumption is implied by assumption A1. Indeed, by Proposition 2.1. of [8], a *necessary* condition for exact controllability at time T of the system (2.1) is that $rankD(t) = n$, $\forall t \in [0,T]$. Then from the Doležal's theorem [31], it follows that there exists the matrix $M(\cdot)$ in assumption A2. In this case, if assumption A2 is not satisfied, then neither will assumption A1, and thus the admissible set $\mathcal{U}_\xi$ will be empty for some ξ and the minimum-energy control problem will not have a solution. One can still formulate a minimum-energy control problem with assumptions A1 and A2 not holding, but then the terminal state cannot be *any* $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}^n)$, but must be restricted to some subset of $L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}^n)$. In this paper, we focus only on the case when ξ can be *any* random variable from the set $L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}^n)$.

We now reformulate the minimum-energy control problem in a more convenient form. Let the processes $z(\cdot) \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^n)$ and $v(\cdot) \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^{m-n})$ be such that

$$u(t) = M(t)\begin{bmatrix} z(t) \\ v(t) \end{bmatrix}. \tag{2.3}$$

Let the matrices $G(\cdot) \in L^\infty(0,T;\mathbb{R}^{n\times n})$, $F(\cdot) \in L^\infty(0,T;\mathbb{R}^{n\times(m-n)})$, $H_1(\cdot) \in L^\infty(0,T;\mathbb{R}^{n\times n})$, $H_2(\cdot) \in L^\infty(0,T;\mathbb{R}^{n\times(m-n)})$, $H_3(\cdot) \in L^\infty(0,T;\mathbb{R}^{(m-n)^2})$, be such that

$$\begin{aligned} B(t)M(t) &= \begin{bmatrix} G(t) & F(t) \end{bmatrix}, \\ M'(t)R(t)M(t) &= \begin{bmatrix} H_1(t) & H_2(t) \\ H_2'(t) & H_3(t) \end{bmatrix}. \end{aligned} \tag{2.4}$$

Due to the symmetric nature of the matrix $R(\cdot)$, the matrices $H_1(\cdot)$ and $H_3(\cdot)$ are also symmetric. Moreover, due to the positive definiteness of $R(\cdot)$ and the Schur's lemma, it holds that

$$\begin{aligned} &H_3(t) > 0, \quad \text{a.e. } t \in [0,T], \\ &H_1(t) - H_2'(t)H_3^{-1}(t)H_2(t) > 0, \quad \text{a.e. } t \in [0,T]. \end{aligned}$$

Eq. (2.1) and the cost functional (2.2) can now be written as

$$\begin{cases} dx(t) = [A(t)x(t) + F(t)v(t) + G(t)z(t)]dt \\ \qquad + [C(t)x(t) + z(t)]dW(t), \\ x(0) = x_0 \in \mathbb{R}^n, \quad \text{is given,} \end{cases} \tag{2.5}$$

$$\begin{aligned} J(v(\cdot), z(\cdot)) = \mathbb{E}\int_0^T &[z'(t)H_1(t)z(t) + 2v'(t)H_2'(t)z(t) \\ &+ v'(t)H_3(t)v(t)]dt. \end{aligned} \tag{2.6}$$

To each element of the set $\mathcal{U}_\xi$ it corresponds a pair of processes $(v(\cdot), z(\cdot))$ from the set

$$\begin{aligned} \mathcal{A}_\xi \equiv \{&v(\cdot) \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^{m-n}), z(\cdot) \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^n) : \\ &x(T) = \xi \text{ a.s.}\}. \end{aligned}$$

In this reformulation, the minimum-energy control problem is:

$$\begin{cases} \min\limits_{(v(\cdot),z(\cdot))\in\mathcal{A}_\xi} J(v(\cdot), z(\cdot)), \\ \text{s.t. (2.5).} \end{cases} \tag{2.7}$$

Before we proceed to its solution, let us state a useful necessary and sufficient condition for the exact controllability of (2.5). It is a slight modification of the result in [29], and we thus omit the proof.

Proposition 1. *Let $E(\cdot) \in L^\infty(0,T;\mathbb{R}^{m\times m})$ be any symmetric matrix such that $E(t) > 0$, a.e. $t \in [0,T]$. Also let $\Phi(\cdot)$ be the unique solution to the equation*

$$\begin{cases} d\Phi(t) = -\Phi(t)[A(t) - G(t)C(t)]dt - \Phi(t)G(t)dW(t), \\ \Phi(0) = I. \end{cases}$$

The system (2.5) is exactly controllable at time T if and only if

$$rank\left[\mathbb{E}\int_0^T \Phi(t)F(t)E(t)F'(t)\Phi'(t)dt\right] = n. \tag{2.8}$$