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# State and output feedback boundary control for a coupled PDE–ODE system ☆

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ABSTRACT

This note is devoted to stabilizing a coupled PDE–ODE system with interaction at the interface. First, a state feedback boundary controller is designed, and the system is transformed into an exponentially stable PDE–ODE cascade with an invertible integral transformation, where PDE backstepping is employed. Moreover, the solution to the resulting closed-loop system is derived explicitly. Second, an observer is proposed, which is proved to exhibit good performance in estimating the original coupled system, and then an output feedback boundary controller is obtained. For both the state and output feedback boundary controllers, exponential stability analyses in the sense of the corresponding norms for the resulting closed-loop systems are provided. The boundary controller and observer for a scalar coupled PDE–ODE system as well as the solutions to the closed-loop systems are given explicitly.

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## 1. Introduction

In control engineering, topics concerning coupled systems are popular, which have rich physical backgrounds such as coupled electromagnetic, coupled mechanical, and coupled chemical reactions. Many results on controllability of the coupled PDE–PDE systems have been achieved such as in [1–3]. Applicable controllers of state and output feedback for coupled PDE–PDE systems as well as coupled PDE–ODE systems, however, are original areas. As a beginning, control design of cascaded PDE–ODE systems were considered in [4–9], where through decoupling and PDE backstepping, boundary controllers were successfully established.

The system considered in this note is a coupled PDE–ODE system with interaction between the ODE and the PDE. At the interface, the ODE acts back on the PDE at the same time as the PDE acts on the ODE. It models the solid–gas interaction of heat diffusion and chemical reaction, where the interaction occurs at the interface; see Fig. 1. This system is certainly more complex than just a single ODE or a single PDE, and even more complex than a cascade of PDE and ODE, in which only the PDE acts on the ODE, or only the ODE acts on the PDE. Thus, it is needed to overcome some difficulties in control design. Some special techniques and PDE backstepping are used to develop controllers.

This note is organized as follows. In Section 2, the problem is formulated. In Section 3, a state feedback boundary controller is designed to stabilize the coupled PDE–ODE system. In Section 4, an observer, as well as an output feedback boundary controller, is designed. And a scalar coupled PDE–ODE system is given in Section 5 as an example, where the controller, the observer and solutions to the closed-loop systems are obtained explicitly. In Section 6, some comments are made on the coupled PDE–ODE systems.

## 2. Problem formulation

Consider the following coupled PDE–ODE system

$$\dot{X}(t) = AX(t) + Bu_x(0,t) \quad (1)$$

$$u_t(x,t) = u_{xx}(x,t), \quad x \in (0,l) \quad (2)$$

$$u(0,t) = CX(t) \quad (3)$$

$$u(l,t) = U(t) \quad (4)$$

where $X(t) \in \mathbb{R}^n$ is the ODE state, the pair $(A, B)$ is assumed to be stabilizable, $u(x,t) \in \mathbb{R}$ is the PDE state, $C^T$ is a constant vector, and $U(t)$ is the scalar input to the entire system. The coupled system is depicted in Fig. 2. The control objective is to exponentially stabilize the system signal $(X(t), u(x,t))$.

## 3. State feedback controller

Admittedly, if an invertible transformation $(X, u) \mapsto (X, w)$ can be sought to transform the system (1)–(4) into an exponentially stable target system companied with a controller, e.g., the following system

$$\dot{X}(t) = (A + BK)X(t) + Bw_x(0,t) \quad (5)$$

$$w_t(x,t) = w_{xx}(x,t), \quad x \in (0,l) \quad (6)$$

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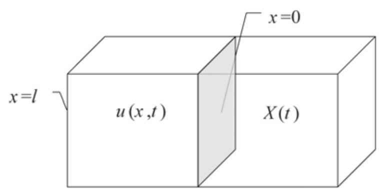


**Fig. 1.** A physical background of the coupled system.

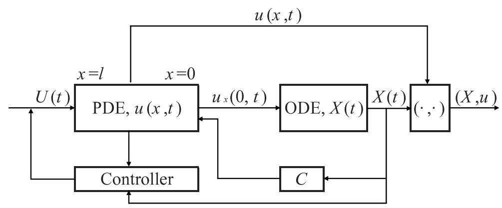


**Fig. 2.** Control configuration of the coupled system.

$$w(0,t) = 0 \tag{7}$$

$$w(l,t) = 0 \tag{8}$$

where $K$ is chosen such that $A + BK$ is Hurwitz, then, exponential stabilization of the original closed-loop system can be achieved. Here the transformation $(X, u) \mapsto (X, w)$ is postulated in the following form

$$X(t) = X(t) \tag{9}$$

$$w(x,t) = u(x,t) - \int_0^x \kappa(x,y)u(y,t)\mathrm{d}y - \Phi(x)X(t) \tag{10}$$

where the gain functions $\kappa(x, y) \in \mathbb{R}$ and $\Phi(x)^T \in \mathbb{R}^n$ are to be determined.

The partial derivatives of $w(x, t)$ with respect to $x$ are given by

$$\begin{aligned} w_x(x,t) &= u_x(x,t) - \kappa(x,x)u(x,t) \\ &\quad - \int_0^x \kappa_x(x,y)u(y,t)\mathrm{d}y - \Phi'(x)X(t) \end{aligned} \tag{11}$$

$$\begin{aligned} w_{xx}(x,t) &= u_{xx}(x,t) - \kappa(x,x)u_x(x,t) \\ &\quad - \left(\frac{\mathrm{d}}{\mathrm{d}x}\kappa(x,x) + \kappa_x(x,x)\right)u(x,t) \\ &\quad - \int_0^x \kappa_{xx}(x,y)u(y,t)\mathrm{d}y - \Phi''(x)X(t) \end{aligned} \tag{12}$$

where the notation $\frac{\mathrm{d}}{\mathrm{d}x}\kappa(x,x) = \kappa_x(x,x) + \kappa_y(x,x)$ is used. The derivative of $w(x, t)$ with respect to $t$ is

$$\begin{aligned} w_t(x,t) &= u_{xx}(x,t) - \kappa(x,x)u_x(x,t) + \kappa_y(x,x)u(x,t) \\ &\quad - \int_0^x \kappa_{yy}(x,y)u(y,t)\mathrm{d}y \\ &\quad + (\kappa(x,0) - \Phi(x)B)u_x(0,t) - \kappa_y(x,0)u(0,t) \\ &\quad - \Phi(x)AX(t). \end{aligned} \tag{13}$$

Setting $x = 0$ in the Eqs. (10) and (11), and by (12) and (13), the following equations hold:

$$w(0,t) = (C - \Phi(0))X(t)$$

$$w_x(0,t) = u_x(0,t) - (\Phi'(0) + \kappa(0,0)C)X(t)$$

$$\begin{aligned} w_t(x,t) - w_{xx}(x,t) &= 2\left(\frac{\mathrm{d}}{\mathrm{d}x}\kappa(x,x)\right)u(x,t) \\ &\quad + \int_0^x (\kappa_{xx}(x,y) - \kappa_{yy}(x,y))u(y,t)\mathrm{d}y \\ &\quad + (\kappa(x,0) - \Phi(x)B)\,u_x(0,t) \\ &\quad + (\Phi''(x) - \Phi(x)A - \kappa_y(x,0)C)X(t) \end{aligned}$$

where the fact that $u(0,t) = CX(t)$ is used. To satisfy (5)–(7), it is sufficient that $\kappa(x, y)$ and $\Phi(x)$ satisfy

$$\kappa_{xx}(x,y) = \kappa_{yy}(x,y) \tag{14}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\kappa(x,x) = 0, \qquad \kappa(x,0) = \Phi(x)B \tag{15}$$

and

$$\Phi''(x) - \Phi(x)A - \kappa_y(x,0)C = 0 \tag{16}$$

$$\Phi(0) = C, \qquad \Phi'(0) = K - \kappa(0,0)C. \tag{17}$$

Although the PDE (14)–(15) and the ODE (16)–(17) are still coupled, they can be decoupled and solved explicitly through some techniques of algebra and analytical mathematics.

First, the solution to the PDE (14)–(15) is

$$\kappa(x,y) = \Phi(x-y)B. \tag{18}$$

Second, substituting (18) into (16) and (17) respectively, it is obtained that

$$\Phi''(x) + \Phi'(x)BC - \Phi(x)A = 0 \tag{19}$$

and

$$\Phi'(0) = K - \Phi(0)BC = K - CBC.$$

Let $I$ be an identity matrix, then the explicit solution to the ODE (16)–(17) is obtained:

$$\Phi(x) = \begin{pmatrix} C & K - CBC \end{pmatrix} e^{Dx} \begin{pmatrix} I \\ 0 \end{pmatrix}$$

where

$$D = \begin{pmatrix} 0 & A \\ I & -BC \end{pmatrix}$$

and the explicit solution to the PDE (14)–(15) is

$$\kappa(x,y) = \begin{pmatrix} C & K - CBC \end{pmatrix} e^{D(x-y)} \begin{pmatrix} I \\ 0 \end{pmatrix} B.$$

The integral transformation $(X, u) \mapsto (X, w)$ defined by (9)–(10) is invertible. Suppose the inverse transformation $(X, w) \mapsto (X, u)$ as the following form

$$X(t) = X(t) \tag{20}$$

$$u(x,t) = w(x,t) + \int_0^x \iota(x,y)w(y,t)\mathrm{d}y + \Psi(x)X(t) \tag{21}$$

where the kernel functions $\iota(x, y) \in \mathbb{R}$ and $\Psi(x)^T \in \mathbb{R}^n$ are to be determined.

Following the same procedure of determination of the kernels $\kappa(x, y)$ and $\Phi(x)$, compute the derivatives $u_x$, $u_{xx}$ and $u_t$, and a sufficient condition for $\iota(x, y)$ and $\Psi(x)$ to satisfy (1)–(3) is obtained as

$$\iota_{xx}(x,y) = \iota_{yy}(x,y) \tag{22}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\iota(x,x) = 0, \qquad \iota(x,0) = \Psi(x)B \tag{23}$$

and

$$\Psi''(x) - \Psi(x)(A + BK) = 0 \tag{24}$$

$$\Psi(0) = C, \qquad \Psi'(0) = K. \tag{25}$$

This cascade system can also be solved explicitly. First, the explicit solution to the ODE (24)–(25) is

$$\Psi(x) = \begin{pmatrix} C & K \end{pmatrix} e^{Ex} \begin{pmatrix} I \\ 0 \end{pmatrix}$$

where

$$E = \begin{pmatrix} 0 & A + BK \\ I & 0 \end{pmatrix}.$$

Second, the explicit solution to the PDE (22)–(23) is

$$\iota(x,y) = \Psi(x-y)B = \begin{pmatrix} C & K \end{pmatrix} e^{E(x-y)} \begin{pmatrix} I \\ 0 \end{pmatrix} B.$$

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