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Internal stabilization of the Oseen–Stokes equations by Stratonovich noise

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A R T I C L E I N F O

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The design of a Stratonovich noise feedback controller with support in an arbitrary open subset $\mathcal{O}_0$ of $\mathcal{O}$ is described. This exponentially stabilizes in probability, that is with probability one, the Oseen–Stokes systems in a domain $\mathcal{O} \subset \mathbb{R}^d$, $d = 2, 3$. This completes the stabilization results from the author's work Barbu (2011) [3] which is concerned with design of an Ito noise stabilizing controller.

1. Introduction and statement of the problem

This work is concerned with the internal stabilization via Stratonovich noise feedback controller of the Oseen–Stokes system

$$\begin{aligned} &\frac{\partial X}{\partial t} - \nu \Delta X + (f \cdot \nabla)X + (X \cdot \nabla)g = \nabla p \quad \text{in } (0, \infty) \times \mathcal{O}, \\ &\nabla \cdot X = 0 \quad \text{in } (0, \infty) \times \mathcal{O}, \\ &X(0, \xi) = x(\xi), \quad \xi \in \mathcal{O},\ X = 0 \text{ on } (0, \infty) \times \partial\mathcal{O}. \end{aligned} \tag{1}$$

Here, $\mathcal{O}$ is an open and bounded subset of $\mathbb{R}^d$, $d = 2, 3$, with smooth boundary $\partial\mathcal{O}$, and $f, g \in C^2(\overline{\mathcal{O}}; \mathbb{R}^d)$ are given functions. In the special case $g \equiv 0$, system (1) describes the dynamics of a fluid Stokes flow with partial inclusion of convection acceleration $(f \cdot \nabla)X$ (X is the velocity field). The same equation describes the disturbance flow induced by a moving body with velocity f through the fluid. We should mention also that in the special case $f \equiv g \equiv X_e$, where X_e is the equilibrium (steady-state) solution of the Navier–Stokes equation,

$$\begin{aligned} &\frac{\partial X}{\partial t} - \nu \Delta X + (X \cdot \nabla)X = \nabla p + f_e, \\ &\nabla \cdot X = 0, \quad X|_{\partial\mathcal{O}} = 0, \end{aligned} \tag{2}$$

and $f_e \in C(\overline{\mathcal{O}}; \mathbb{R}^d)$, system (1) is the linearization of (2) around X_e. In this way, the stabilization of (1) can be interpreted as the first order stabilization procedure of steady-state Navier–Stokes flows.

Our aim here is to design a stochastic feedback controller of the form

$$u = \mathbf{1}_{\mathcal{O}_0} \sum_{k=1}^{M} R_k(X) \circ \dot{\beta}_k, \quad R_k \in L((L^2(\mathcal{O}))^d), \tag{3}$$

which stabilizes in probability system (1) and has support in an open subdomain $\mathcal{O}_0$ of $\mathcal{O}$.

Here, $\{\beta_k\}_{k=1}^M$ is a system of mutually independent Brownian motions in a probability space $\{\Omega, \mathcal{F}, \mathbb{P}\}$ with filtration $\{\mathcal{F}_t\}_{t>0}$ while the corresponding closed loop system

$$\begin{aligned} &dX - \nu \Delta X\, dt + (f \cdot \nabla)X\, dt + (X \cdot \nabla)g\, dt \\ &\quad = \mathbf{1}_{\mathcal{O}_0} \sum_{k=1}^{M} R_k(X) \circ d\beta_k + \nabla p\, dt \\ &X(0) = x \end{aligned} \tag{4}$$

is taken in the Stratonovich sense (see, e.g., [1]) and this is the significance of the symbol $R_k(X) \circ \dot{\beta}_k$ in the expression of the noise controller (3). In the following, the terminology Stratonovich feedback controller or Ito feedback controller refer to the sense in which the corresponding stochastic equation (4) is considered: in the Stratonovich sense or, respectively, the Ito sense. We have denoted by $\mathbf{1}_{\mathcal{O}_0}$ the characteristic function of the open set $\mathcal{O}_0 \subset \mathcal{O}$.

In [2–5], the author has designed similar stabilizable Ito noise controllers for Eq. (1) and related Navier–Stokes equations. However, it should be said that, with respect to Ito noise controllers, the Stratonovich feedback controller (3) has the advantage of being stable with respect to smooth changes $\dot{\beta}_k^{\varepsilon}$ of the noise $\dot{\beta}_k$, and this fact is crucial not only from the conceptual point of view, but also for numerical simulations

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and practical implementation into system (1) of the random stabilizable feedback controller

$$u_\varepsilon(t) = \mathbf{1}_{\mathcal{O}_0} \sum_{k=1}^{M} R_k(X(t))\dot{\beta}_k^\varepsilon(t),$$

where $\dot{\beta}_k^\varepsilon$ is a smooth approximation of β_k^ε. If instead (3) we take u to be an Ito stabilizable feedback controller, then the corresponding Ito stochastic closed loop equation is convergent for $\varepsilon \to 0$ to a Stratonovich equation of the form (4) which might be unstable because the stabilization effect of the noise controller is given by the Ito's formula which is valid for Ito stochastic equation only. Thus, for numerical implementations of a stabilizanble noise feedback controller u it is essential that it is of Stratonovich type. It should be said, however, that the option for an Ito stabilizable noise controller, as in [2–5], or a Stratonovich one, as in this work, is a function of the specific technique used to insert such a noise controller into the system (1), i.e. by direct simulation or by a numerical approximation procedure.

As regards the literature on stabilization of linear differential systems by Stratonovich noise, the pioneering works [6,7] should be primarily cited. For linear PDEs, this procedure was developed in [8,9] which are related to this work. For general results on internal stabilization of Navier–Stokes systems with deterministic feedback controllers, we refer to [10]. (See also [11] for noise stabilization effect in a different nonlinear PDE context.)

2. The noise stabilizing feedback controller

Consider the standard space of free divergence vectors $H = \{y \in (L^2(\mathcal{O}))^d; \nabla \cdot y = 0 \text{ in } \mathcal{O}, y \cdot n = 0 \text{ on } \partial\mathcal{O}\}$ and denote by $\mathcal{A}_0 : D(\mathcal{A}_0) \subset H \to H$ the realization of the Oseen–Stokes operator in this space, that is,

$$\mathcal{A}_0 y = P(-\nu\Delta y + (f \cdot \nabla)y + (y \cdot \nabla)g), \quad y \in D(\mathcal{A}_0), \tag{5}$$

where $D(\mathcal{A}_0) = H \cap (H^2(\mathcal{O}))^d \cap (H_0^1(\mathcal{O}))^d$. Here, P is the Leray projector on H, and $H^2(\mathcal{O}), H_0^1(\mathcal{O})$ are standard Sobolev spaces on $\mathcal{O}$. In the following, it will be more convenient to represent Eq. (1) in the complex Hilbert space $\mathcal{H} = H + iH$ by extending $\mathcal{A}_0$ to $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ via the standard procedure, $\mathcal{A}(x + iy) = \mathcal{A}_0 x + i\mathcal{A}_0 y$. The operator $\mathcal{A}$ has a countable set of eigenvalues $\{\lambda_j\}_{j=1}^\infty$ (eventually complex) with the corresponding eigenvectors φ_j. Denote by $\mathcal{A}^*$ the adjoint of $\mathcal{A}$ with eigenvalues $\overline{\lambda}_j$ and eigenvectors φ_j^*. Each eigenvalue is repeated in the following according to its algebraic multiplicity m_j. Normalizing the system $\{\varphi_j\}_{j=1}^\infty$, we see that

$$|\nabla\varphi_j|_{\mathcal{H}}^2 = \lambda_j - \left\langle (f \cdot \nabla)\varphi_j + (\varphi_j \cdot \nabla)g, \varphi_j \right\rangle, \quad \forall j,$$

and since, by the Fredholm–Riesz theory, $|\lambda_j| \to +\infty$ as $j \to \infty$, we infer that $\operatorname{Re}\lambda_j \to +\infty$ as $j \to \infty$. We denote by N the minimal number of eigenvalues λ_j for which

$$\operatorname{Re}\lambda_j > 0 \quad \text{for } j \geq N,\ \lambda_1 + \lambda_2 + \cdots + \lambda_N > 0. \tag{6}$$

(In the above sequence, each λ_j is taken together its conjugate $\overline{\lambda}_j$ and, clearly, there is such a natural number N.)

Set $\mathcal{X}_u = \operatorname{lin\,span}\{\varphi_j\}_{j=1}^N$ and denote by $\mathcal{X}_s$ the algebraic complement of $\mathcal{X}_u$ in $\mathcal{X}$. It is well known that $\mathcal{X}_u$ and $\mathcal{X}_s$ are both invariant for $\mathcal{A}$ and, if we set

$$\mathcal{A}_u = \mathcal{A}|_{\mathcal{X}_u}, \qquad \mathcal{A}_s = \mathcal{A}|_{\mathcal{X}_s},$$

we have for their spectra $\sigma(\mathcal{A}_u) = \{\lambda_j\}_{j=1}^N$, $\sigma(\mathcal{A}_s) = \{\lambda_j\}_{j=N+1}^\infty$ and, since $-\mathcal{A}_s$ is the generator of an analytic C-semigroup $e^{-\mathcal{A}_s t}$ in $\mathcal{X}_s$, we have

$$\|e^{-\mathcal{A}_s t}\|_{L(\mathcal{H})} \leq C\exp(-\operatorname{Re}\lambda_{N+1}t), \quad t \geq 0, \tag{7}$$

(see, e.g., [2, p. 14]). In the following, we shall assume that

(i) *All the eigenvalues* $\{\lambda_j\}_{j=1}^N$ *are semisimple.*

This means that the algebraic multiplicity of each λ_j, $j = 1, \ldots, N$, coincides with its geometric multiplicity or, in other words, the finite-dimensional operator (matrix) $\mathcal{A}_u$ can be diagonalized. As we will see later on, this assumption is not essentially necessary but it simplifies the construction of the stabilizing controller because it reduces the unstable part of the system to a diagonal finite-dimensional differential system. In particular, it follows by (i) that we can choose the dual systems $\{\varphi_j\}$ and $\{\varphi_j^*\}$ in such a way that

$$\langle \varphi_i, \varphi_j^* \rangle = \delta_{ij}, \quad i, j = 1, \ldots, N. \tag{8}$$

(Here, and everywhere in the following, $\langle \cdot, \cdot \rangle$ stands for the scalar product in $\mathcal{H}$ and H. By $|\cdot|_{\mathcal{H}}$ and $|\cdot|_H$ we denote the corresponding norms.)

We note that the uncontrolled Oseen–Stokes system (1) can be rewritten in the space $\mathcal{H}$ as

$$\frac{dX}{dt} + \mathcal{A}X = 0, \quad t \geq 0,\ X(0) = x, \tag{9}$$

and setting $X_u = \sum_{j=1}^N y_j\varphi_j$, $X_s = (I - P_N)X$, where P_N is the algebraic projector on $\mathcal{X}_u$, we have

$$\frac{dX_u}{dt} + \mathcal{A}_u X_u = 0, \quad X_u(0) = P_N x, \tag{10}$$

$$\frac{dX_s}{dt} + \mathcal{A}_s X_s = 0, \quad X_s(0) = (I - P_N)x. \tag{11}$$

We set $A_u = \left\{\langle \mathcal{A}\varphi_j, \varphi_k^* \rangle\right\}_{j,k=1}^N = \operatorname{diag}\|\lambda_j\|_{j=1}^N$ and so, by (8), we may rewrite (10) in terms of $y = \left\{y_j = \langle X_u, \varphi_j^* \rangle\right\}_{j=1}^N$ as

$$\frac{dy}{dt} + A_u y = 0, \quad y(0) = \left\{\langle P_N x, \varphi_j^* \rangle\right\}_{j=1}^N. \tag{12}$$

Since $\operatorname{Tr}(-A_u) = -\lambda_1 - \lambda_2 - \cdots - \lambda_N < 0$, it follows by Theorem 2 in [7] that there is a sequence of skew-symmetric matrices $\{C^k\}_{k=1}^M$, where $M = N - 1$ such that the solution y to the Stratonovich stochastic system

$$dy + A_u y\,dt = \sum_{k=1}^{M} C^k y \circ d\beta_k, \quad t \geq 0, \tag{13}$$

has the property

$$|y(t)| \leq C|y(0)|e^{-\gamma_0 t}, \quad \mathbb{P}\text{-a.s., } \forall t > 0, \tag{14}$$

where $\gamma_0 > 0$. The matrix C^k is explicitly constructed in [7] and it will be used below to construct a stabilizable feedback controller of the form (3). Namely, we set in (3)

$$R_k(X) = \sum_{i,j=1}^{N} C_{ij}^k \langle X, \varphi_j^* \rangle \phi_i, \quad k = 1, \ldots, M, \tag{15}$$

where $\|C_{ij}^k\|_{i,j=1}^N = C^k$,

$$\phi_i = \sum_{\ell=1}^{N} \alpha_{i\ell}\varphi_\ell^*, \quad i = 1, \ldots, N, \tag{16}$$

and $\alpha_{i\ell}$ are chosen in such a way that

$$\sum_{\ell=1}^{N} \alpha_{i\ell}\gamma_{\ell j} = \delta_{ij}, \quad i, j = 1, \ldots, N. \tag{17}$$

Here, $\gamma_{\ell j} = \int_{\mathcal{O}_0} \varphi_\ell^* \overline{\varphi}_j^* d\xi$ and since, by the unique continuation property (see [2, p. 157]), the eigenfunction system $\{\varphi_j^*\}$ is linearly independent on $\mathcal{O}_0$, we infer that the matrix $\|\gamma_{\ell j}\|_{\ell,j=1}^N$ is not singular and, therefore, there is a unique system $\{\alpha_{i\ell}\}$ which satisfies (17). Then, by (16), we see that