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The ergodic property and positive recurrence of a multi-group Lotka–Volterra mutualistic system with regime switching

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ABSTRACT

In this paper, we consider a stochastic multi-group Lotka–Volterra mutualistic system under regime switching. It is well known that the population is forced to expire when the perturbation is sufficiently large. The main aim here is to investigate its ergodic property and positive recurrence by stochastic Lyapunov functions under small perturbation, which can be used to explain some recurring phenomena in practice and thus provide a good description of permanence. The mean of the stationary distribution is estimated. Simulations are also carried out to confirm our analytical results.

1. Introduction

Taking the white and color noise into account, population systems described by stochastic differential equations with regime switching have recently been studied by many authors; see [1–9], for example. It is well known that, when the perturbation is large, the population will be forced to expire whilst it remains stochastic permanent when the perturbation is small, which can provide some explanation of dynamical behavior from a biological perspective.

In practice, we may often observe the recurrence of higher and lower population levels of a permanent population system as time goes by. See [10–18] and their references for pollination mutualism as examples. If we make a great number of records to investigate the dynamic behavior of a population system, we may find that the average of the records approaches a fixed positive point, but a single record may fluctuate around this fixed point even if the number of records is large. Then, how can we explain such biological phenomena? In such a case, stochastic permanence or limits of integral average do not seem adequate, so we need to investigate other dynamical properties to illustrate such biological phenomena. Therefore, in this paper, we concentrate on the ergodic property and positive recurrence of a multi-group Lotka–Volterra mutualistic population system to try to give a good explanation of the above biological phenomena (see Remark 3.1).

Consider a stochastic multi-group Lotka–Volterra system characterized by the following stochastic differential equation with color and white noise:

$$\begin{aligned} dx(t) &= \operatorname{diag}(x_1(t), \ldots, x_n(t))[(b(r(t)) + A(r(t))x(t)dt) \\ &\quad + \sigma(r(t))dB(t)], \end{aligned} \tag{1.1}$$

where $x = (x_1, \ldots, x_n)^\tau$, $B = (B_1, \ldots, B_d)^\tau$ is a standard d-dimensional Brownian motion, $\{r(t), t \geq 0\}$ is a right-continuous Markov chain independent of the Brownian motion B, taking values in a finite state space $\mathbb{S} = \{1, \ldots, N\}$ with generator $\Gamma = (\gamma_{ij})_{N\times N}$ given by

$$\mathbb{P}\{r(t+\delta) = j | r(t) = i\} = \begin{cases} \gamma_{ij}\delta + o(\delta) & \text{if } i \neq j, \\ 1 + \gamma_{i,i}\delta + o(\delta) & \text{if } i = j, \end{cases} \tag{1.2}$$

and, for any $k \in \mathbb{S}$, $b(k) = (b_1(k), \ldots, b_n(k))^\tau$, $A(k) = (a_{ij}(k))_{n\times n}$, $\sigma(k) = (\sigma_{ij}(k))_{n\times d}$. We also assume that, for $k \in \mathbb{S}$, $\operatorname{Rank}(\sigma(k)) = n$, and $a_{ii}(k) < 0$, $a_{ij}(k) \geq 0$, $1 \leq i, j \leq n$, $i \neq j$. This means that Eq. (1.1) is a mutualistic system in which every species enhances the growth of each other [19–21].

In this paper, we investigate the ergodic property and positive recurrence of Eq. (1.1) by stochastic Lyapunov functions with regime switching [22], which implies the existence and uniqueness of a stationary distribution. We show that the ergodic property and positive recurrence can provide a biological perspective of cycling phenomena of a population system, and hence describe the permanence of a population system in practice.

The paper is organized as follows. In Section 2, we introduce some notation and assumptions, which are necessary for later discussion. In Section 3, we use a class of stochastic Lyapunov functions with regime switching to obtain the ergodic property and

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positive recurrence, which account for some recurring events of a population system. The mean of the stationary distribution of a population system is also investigated. In Section 4, we report results of computer simulations to confirm our theoretical analysis. In Section 5, we provide a concluding discussion to end this paper.

2. Preliminaries

Throughout this paper, unless otherwise specified, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is right continuous and increasing while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). R^n_+ denotes the positive zone in R^n, i.e., $R^n_+ = \{x \in R^n; x_i > 0, 1 \leq i \leq n\}$. If B is a symmetric $n \times n$ matrix, we recall the following notation:

$$\lambda^+_{\max}(B) = \sup_{x\in R^n_+, |x|=1} x^\tau Bx, \tag{2.1}$$

which is introduced in [1,23]. Thus, for any $x \in R^n_+$, we have

$$x^\tau Bx \leq \lambda^+_{\max}(B)|x|^2. \tag{2.2}$$

We also assume that Markov chain $\{r(t), t \geq 0\}$ is irreducible, i.e., each state can reach any other in finite time. By the classical theory of a Markov chain, the finite states imply the ergodicity and positive recurrence of $\{r(t), t \geq 0\}$. Hence Markov chain $\{r(t), t \geq 0\}$ has a unique stationary distribution $\pi = (\pi_1, \dots, \pi_N)$ such that

$$\pi\Gamma = 0, \qquad \sum_{k=1}^{N} \pi_k = 1, \quad \pi_k > 0, 1 \leq k \leq N \tag{2.3}$$

and, for any vector $f = (f(1), \dots, f(N))^\tau$,

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f(r(t))\mathrm{d}t = \sum_{i=1}^{N} f(i)\pi_i.$$

Now, we introduce some criteria on the ergodic property and positive recurrence of diffusion systems with a switching regime [22]. Let $(y(t), r(t))$ be the diffusion process described by the following equation:

$$\begin{aligned} \mathrm{d}y(t) &= f(y(t), r(t))\mathrm{d}t + g(y(t), r(t))\mathrm{d}B(t), \\ y(0) &= y, \qquad r(0) = r, \end{aligned} \tag{2.4}$$

where $B(\cdot)$ and $r(\cdot)$ are the d-dimensional Brownian motion and the right-continuous Markov chain in the above discussion, respectively, and $f(\cdot,\cdot) : R^n \times \mathbb{S} \to R^n$, $g(\cdot,\cdot) : R^n \times \mathbb{S} \to R^{n\times d}$ satisfying $g(y, k)g^\tau(y, k) = C(y, k)$, $1 \leq k \leq n$. For each $k \in \mathbb{S}$, and for any twice continuously differentiable function $V(\cdot, k)$, we define $\mathcal{L}$ by

$$\begin{aligned} \mathcal{L}V(y, k) = &\sum_{i=1}^{n} f_i(y, k)\frac{\partial V(y, k)}{\partial y_i} \\ &+ \frac{1}{2}\sum_{i,j=1}^{n} C_{ij}(y, k)\frac{\partial^2 V(y, k)}{\partial y_i \partial y_j} + \sum_{i=1}^{N}\gamma_{ki}V(y, i). \end{aligned}$$

To show the ergodic property and positive recurrence of Eq. (1.1), we introduce some well-known results (implied by Theorem 3.10 on p. 1162, Theorem 3.13 on p. 1164, Theorem 4.3 on p. 1168, and Theorem 4.4 on p. 1169 of [22]) as a lemma.

Lemma 2.1. *If the following conditions are satisfied:*

(i) *for any $k \in \mathbb{S}$, $C(y, k)$ is symmetric and satisfies*

$$\kappa_1|\xi|^2 \leq \langle C(y, k)\xi, \xi\rangle \leq \kappa_1^{-1}|\xi|^2 \quad \text{for all } \xi \in R^n,$$

with some constant $\kappa_1 \in (0, 1]$ for all $y \in R^n$;

(ii) *for all $i \neq j$, $\gamma_{ij} > 0$;*

(iii) *there exists a nonempty open set D with compact closure, satisfying that, for each $k \in \mathbb{S}$, there exists a nonnegative function $V(\cdot, k) : D^c \to R$ such that $V(\cdot, k)$ is twice continuously differentiable and that, for some $\alpha > 0$,*

$$\mathcal{L}V(y, k) \leq -\alpha, \quad (y, k) \in D^c \times \mathbb{S},$$

then $(y(t), r(t))$ of Eq. (2.4) is ergodic and positive recurrent. That is to say, there exists a unique stationary density $\mu(\cdot,\cdot)$ and, for any Borel measurable function $f(\cdot,\cdot) : R^n \times \mathbb{S} \to R$ such that

$$\sum_{k\in\mathbb{S}} \int_{R^n} |f(y, k)|\mu(y, k)\mathrm{d}y < \infty,$$

we have

$$\mathbf{P}\left\{\frac{1}{T}\int_0^T f(y(t), r(t))\mathrm{d}t \to \sum_{k\in\mathbb{S}}\int_{R^n} f(y, k)\mu(y, k)\mathrm{d}y \text{ as } T \to \infty\right\} = 1.$$

Positive recurrence of $(y(t), r(t))$ means that, for any $\tilde{U} = \tilde{D} \times \{k\}$, where $\tilde{D}$ is any nonempty open set and $k \in \mathbb{S}$, $(y(t), r(t))$ can reach $\tilde{U}$ in finite time, a.e., and their hitting time is integrable (see Section 3 in [22]).

3. Ergodic property of positive recurrence

Before further discussion, we need to show that Eq. (1.1) has a unique positive solution, which is essential in modeling a population system. Since the coefficients of Eq. (1.1) do not satisfy the linear growth condition, the classical theory of stochastic differential equations is not applicable directly. In recent papers (see, e.g., [24,23,25,26]), there are some standard techniques to prove the existence and uniqueness of a positive solution to Eq. (1.1), so we give the following theorem without proof.

Theorem 3.1. *Assume that, for any $k \in \mathbb{S}$, there exist positive constants $c_1(k), \dots, c_n(k)$ such that*

$$\max_{k\in\mathbb{S}} \left\{\lambda^+_{\max}\left(\bar{C}(k)A(k) + A^\tau(k)\bar{C}(k)\right)\right\} \leq 0,$$

where $\bar{C}(k) = \mathrm{diag}(c_1(k), \dots, c_n(k))$. Then, for any initial value $x(0) \in R^n_+$, $r(0) \in \mathbb{S}$, Eq. (1.1) has a unique positive solution almost surely.

It is well known that large perturbation makes a population system expire (see, e.g., [1,24,4]). Then what will happen when the perturbation is relatively small? Recently, Li et al. [24] and Pang et al. [5] use the moment estimate and the Borel–Cantelli lemma to obtain the stochastic permanence and asymptotic bound of the integral average under small perturbation. In particular, Mao [21] discusses the existence and uniqueness of a stationary distribution in the presence of white noise. But there are few papers that have concentrated on the ergodic property and positive recurrence of a population system with a switching regime, which can provide a better description of the permanence of a population system in practice, and illustrate some recurrent events from a biological perspective.

First, by recent papers (see, e.g., [24,23,25,26]), there is the uniqueness of a positive solution to Eq. (1.1), that is to say, $x_i(t) > 0$, $1 \leq i \leq n$. Thus, for any $1 \leq i \leq n$, let

$$u_i(t) = \log x_i(t), \quad \text{for } t \geq 0 \tag{3.1}$$

and $u(t) = (u_1(t), \dots, u_n(t))$. Then Itô's formula yields, for any $1 \leq i \leq n$,

$$\begin{aligned} \mathrm{d}u_i(t) = &\left[b_i(r(t)) + \sum_{j=1}^{n} a_{ij}(r(t))\exp(u_j(t))\right. \\ &\left. - \frac{1}{2}\sum_{j=1}^{d}\sigma^2_{ij}(r(t))\right]\mathrm{d}t + \sum_{j=1}^{d}\sigma_{ij}(r(t))\mathrm{d}B_j(t). \end{aligned} \tag{3.2}$$