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Asymptotic stabilization with locally semiconcave control Lyapunov functions on general manifolds

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ABSTRACT

Asymptotic stabilization on noncontractible manifolds is a difficult control problem. If a configuration space is not a contractible manifold, we need to design a time-varying or discontinuous state feedback control for asymptotic stabilization at the desired equilibrium.

For a system defined on Euclidean space, a discontinuous state feedback controller was proposed by Rifford with a semiconcave strict control Lyapunov function (CLF). However, it is difficult to apply Rifford's controller to stabilization on general manifolds.

In this paper, we restrict the assumption of semiconcavity of the CLF to the "local" one, and introduce the disassembled differential of locally semiconcave functions as a generalized derivative of nonsmooth functions. Further, we propose a Rifford–Sontag-type discontinuous static state feedback controller for asymptotic stabilization with the disassembled differential of the locally semiconcave practical CLF (LS-PCLF) by means of sample stability. The controller does not need to calculate limiting subderivative of the LS-PCLF.

Moreover, we show that the LS-PCLF, obtained by the minimum projection method, has a special advantage with which one can easily design a controller in the case of the minimum projection method. Finally, we confirm the effectiveness of the proposed method through an example.

1. Introduction

Asymptotic stabilization on noncontractible manifolds, a difficult control problem [1–3], has been studied by a few researchers [1,2,4–7]. The main problem is that a noncontractible manifold, as a configuration space, never has a continuous asymptotically stabilizing static state feedback control at any desired equilibrium. Hence one needs to design a discontinuous or time-varying stabilizing controller [3].

Control Lyapunov functions (CLFs) play an important role in feedback control design [3,8,9]. In particular, semiconcave strict CLFs enable designing discontinuous asymptotic stabilizing controllers [10]. Rifford proposed a discontinuous controller, defined on Euclidean space, based on semiconcave strict CLFs [10]. However, Rifford's controller cannot be directly applied to stabilization on manifolds or systems with unbounded inputs.

In this paper, we introduce the framework of a locally semiconcave practical CLF for stabilization on manifolds. We consider the disassembled differential instead of the limiting subderivative of a locally semiconcave function. Then, we show that the directional subderivative used in the definition of the practical CLF is replaced with the disassembled differential. Further, we propose a Rifford–Sontag-type discontinuous asymptotically stabilizing static state feedback controller with the disassembled differential of the locally semiconcave practical control Lyapunov function (LS-PCLF) by means of sample stability.

For general differentiable manifolds, we proposed the minimum projection method to design a locally semiconcave strict CLF [11,12], but we did not show how to stabilize the origin of control systems defined on manifolds with the LS-PCLFs. In this paper, we show that the locally semiconcave CLF, obtained by the minimum projection method, is particularly advantageous for calculating the disassembled differential. Therefore, one can easily design a controller when the LS-PCLF is obtained by the minimum projection method.

2. Preliminaries

2.1. Differentiable manifolds

A brief introduction of differentiable manifolds is necessary to discuss the control systems defined on manifolds [13,14]. In this

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paper, $\mathcal{X}$ denotes an n-dimensional smooth manifold, $T_x\mathcal{X}$ a vector space called the tangent space to $\mathcal{X}$ at x, and an element of $T_x\mathcal{X}$ a tangent vector at x. $T_x^*\mathcal{X}$ denotes the dual space to $T_x\mathcal{X}$, called the cotangent space at x, and an element of $T_x^*\mathcal{X}$ a cotangent vector (or a differential 1-form) at x. A subset $\mathcal{M} \subset \mathcal{X}$ is said to be precompact in $\mathcal{X}$, if its closure in $\mathcal{X}$ is compact. For each $x \in \mathcal{X}$, there exists a local chart $(\mathcal{W}, \eta)$ such that $\mathcal{W} \subset \mathcal{X}$ and $\eta : \mathcal{W} \to \mathcal{Y} = \mathrm{Im}(\eta) \subset \mathbb{R}^n$ is a homeomorphism. Then, $\eta(x)$ is called the local coordinate representation of x with the chart $(\mathcal{W}, \eta)$.

Consider a function $V : \mathcal{X} \to \mathbb{R}$ and a chart $(\mathcal{W}, \eta)$ for $\mathcal{X}$. Then, the function $V_\mathcal{W} : \mathcal{Y} \to \mathbb{R}$, defined by $V_\mathcal{W}(\xi) = V \circ \eta^{-1}(\xi)$, is called the coordinate representation of V. Note that $V_\mathcal{W}$ is defined on a subset of $\mathbb{R}^n$. Therefore, addition and scalar multiplication are defined as usual. By the same manner, $f_\mathcal{W}$ denotes the local coordinate representation of $f \in T_x\mathcal{X}$. Let $V : \mathcal{X} \to \mathbb{R}$ be a smooth function. The mapping $\mathrm{d}V : \mathcal{X} \to T_x^*\mathcal{X}$ denotes the differential of V. Let $(\xi_1, \ldots, \xi_n)$ be local coordinates of $\mathcal{X}$ with a local chart $(\mathcal{W}, \eta)$. Then, the mapping $\mathrm{d}V_\mathcal{W}$ can be defined by

$$\mathrm{d}V_\mathcal{W}(\eta(x)) = \sum_{i=1}^{n} \frac{\partial V_\mathcal{W}}{\partial \xi_i}(\eta(x))\mathrm{d}\xi_i. \tag{1}$$

The natural pairing $\langle \mathrm{d}V, f\rangle$ between a cotangent vector and a tangent vector is defined by Lie derivative as follows: $\langle \mathrm{d}V, f\rangle = L_f V$. In local coordinates,

$$\langle \mathrm{d}V(x), f(x)\rangle_\mathcal{W} = \sum_{i=1}^{n} \frac{\partial V_\mathcal{W}}{\partial \xi_i} f_{\mathcal{W}i}(\eta(x)). \tag{2}$$

If $\mathcal{X}$ and $\tilde{\mathcal{X}}$ are smooth manifolds and $\phi : \tilde{\mathcal{X}} \to \mathcal{X}$ is a smooth mapping, for each $\tilde{x} \in \tilde{\mathcal{X}}$, the mapping $\phi_* : T_{\tilde{x}}\tilde{\mathcal{X}} \to T_{\phi(\tilde{x})}\mathcal{X}$ denotes the differential (or the pushforward) of ϕ. Let $(\xi_1, \ldots, \xi_n)$ and $(\tilde{\xi}_1, \ldots, \tilde{\xi}_n)$ be local coordinates of $\mathcal{X}$ and $\tilde{\mathcal{X}}$ at $\phi(\tilde{x})$ and $\tilde{x}$ with local charts $(\mathcal{W}, \eta)$ and $(\tilde{\mathcal{W}}, \tilde{\eta})$, respectively. Then, the mapping $\phi_{*\tilde{\mathcal{W}},\mathcal{W}} : \tilde{\mathcal{Y}} \to \mathcal{Y}$ can be defined by

$$\phi_{*\tilde{\mathcal{W}},\mathcal{W}}\left(\frac{\partial}{\partial \tilde{\xi}_i}\right) = \sum_{j=1}^{n} \frac{\partial(\eta \circ \phi \circ \tilde{\eta}^{-1})_j}{\partial \tilde{\xi}_i} \frac{\partial}{\partial \xi_j} \quad (1 \le i \le n). \tag{3}$$

A function $V : \mathcal{X} \to \mathbb{R}$ is called locally semiconcave (with linear modulus [15]) if for any chart $(\mathcal{W}, \eta)$ and compact set $\mathcal{M} \subset \mathcal{W}$, there exists $C > 0$ such that

$$V(x) + V(y) - 2V_\mathcal{W}\left(\frac{1}{2}(\eta(x) + \eta(y))\right) \le C\|\eta(x) - \eta(y)\|^2 \tag{4}$$

for all $x, y \in \mathcal{M}$ satisfying $(\eta(x) + \eta(y))/2 \in \eta(\mathcal{M})$. Note that, the existence of C does not depend on the choice of charts.

2.2. Control systems defined on differentiable manifolds

We consider the following nonlinear control system on a finite-dimensional arc-connected C^1-differentiable manifold $\mathcal{X}$:

$$\dot{x} = f(x, u), \tag{5}$$

where $x \in \mathcal{X}, u \in U \subset F(\mathbb{R}, \mathbb{R}^m); t \mapsto u(t) \in \mathcal{U} \subset \mathbb{R}^m$, and where $F(\mathbb{R}, \mathbb{R}^m)$ denotes a set of mappings from $\mathbb{R}$ to $\mathbb{R}^m$. Moreover, a mapping $f : \mathcal{X} \times \mathcal{U} \to T_x\mathcal{X}$ is assumed to satisfy $f(0, 0) = 0$, where $0 \in \mathcal{X}$, called the origin, is the desired equilibrium, and locally Lipschitz continuous with respect to x; i.e., for a fixed $u_0 \in \mathcal{U}$, a local chart $(\mathcal{W}, \eta)$ and a compact set $\mathcal{M} \subset \mathcal{W}$, there exists L such that

$$\|f_\mathcal{W}(\eta(y), u_0) - f_\mathcal{W}(\eta(x), u_0)\| < L\|\eta(y) - \eta(x)\| \tag{6}$$

for all $x, y \in \mathcal{M}$.

A function $k : \mathcal{X} \to U$ is called a static state feedback (or simply feedback). The objective of the paper is to develop an asymptotically stabilizing static state feedback controller $u = k(x)$ at the origin of (5).

We consider the sample-and-hold solution, defined as follows, as solutions of (5).

Definition 1 (*Partition [5,16]*). Any infinite sequence $\pi = \{t_i \in \mathbb{R}_{\ge 0}\}_{i\in\mathbb{Z}_{\ge 0}}$ consisting of numbers $0 = t_0 < t_1 < t_2 < \cdots$ with $\lim_{i\to\infty} t_i = +\infty$ is called a partition and the number $d(\pi) := \sup_{i\in\mathbb{Z}_{\ge 0}}(t_{i+1} - t_i)$ its diameter.

Definition 2 (*Sample-and-Hold Solution [5,16,17]*). Let $u = k(x)$ be a given feedback, π a partition, and $x \in \mathcal{X}$ an initial state. The sample-and-hold solution $\psi(t, x, k(x)) : \mathbb{R}_{\ge 0} \times \mathcal{X} \times U \to \mathcal{X}$ for (5) is defined as the mapping such that $\psi(t, x, k(x)) = x(t)$, where $x(t)$ is a continuous mapping obtained by recursively solving

$$\dot{x}(t) = f(x(t), k(x(t_i))) \tag{7}$$

from the initial time t_i to the maximal time

$$s_i = \max\{t_i, \sup\{s \in [t_i, t_{i+1}]|x(\cdot) \text{ is defined on } [t_i, s)\}\}, \tag{8}$$

with $x(0) = x$.

The feedback $u = k(x)$ implicitly determines the control $u(t) = k(x(t))$ by the sample-and-hold solution. Note that every sample-and-hold solution is absolutely continuous. Then, the following lemma holds:

Lemma 1. *Consider a diffeomorphism $\phi : \tilde{\mathcal{X}} \to \mathcal{M}$, where $\mathcal{M} \subset \mathcal{X}$. Then, $\phi^{-1}(\psi(t, x, k(x)))$ is a sample-and-hold solution of $\dot{\tilde{x}} = \phi_*^{-1} f(\phi(\tilde{x}), k(\phi(\tilde{x})))$ if and only if $\psi(t, x, k(x))$ is a sample-and-hold solution of (5) on $\mathcal{M}$.*

We define sample stability as follows [5, s-stability]:

Definition 3 (*Sample Stability*). Consider system (5). $\mathfrak{P}$ denotes the set of all open precompact subset of $\mathcal{X}$ containing the origin.

A feedback $k : \mathcal{X} \to \mathcal{U}$ is said to sample stabilize the origin of the system (5) if the following holds for arbitrary sets $\mathcal{R}_1, \mathcal{R}_2 \in \mathfrak{P}$ such that $\mathcal{R}_1 \subset \mathcal{R}_2$.

(1) There exists a set $\mathcal{M} \subset \mathcal{X}$ depending only upon $\mathcal{R}_2$ and two positive numbers $\Omega, T > 0$ depending on $\mathcal{R}_1$ and $\mathcal{R}_2$ such that, for any initial value $x \in \mathcal{R}_2$, for any partition π of the diameter less than Ω, the corresponding sample-and-hold solution $\psi(t, x, k(x))$ satisfies the following conditions:
    (a) $\psi(t, x, k(x)) \in \mathcal{R}_1$ for all $t \ge T$,
    (b) $\psi(t, x, k(x)) \in \mathcal{M}$ for all $t \ge 0$.
(2) for each $\mathcal{E} \in \mathfrak{P}$, there exists a set $\mathcal{P} \in \mathfrak{P}$ such that if $\mathcal{R}_2 \subset \mathcal{P}$, $\mathcal{M}$ in (1) can be chosen satisfying $\mathcal{M} \subset \mathcal{E}$.

3. Locally semiconcave control Lyapunov functions

3.1. Locally semiconcave practical control Lyapunov functions

Strict CLFs are commonly used for the development of an asymptotically stabilizing controller. For discontinuous control design, semiconcave strict CLF was introduced by Rifford [10]. The locally semiconcave strict CLF is defined as follows.

Definition 4 (*Locally Semiconcave Strict CLF*). A global locally semiconcave strict control Lyapunov function for system (5) is a locally semiconcave function $V : \mathcal{X} \to \mathbb{R}$ such that the following properties hold:

(A1) V is proper; that is, the set $\{x \in \mathcal{X}|V(x) \le L\}$ is compact for every $L > 0$.
(A2) V is positive definite; that is, $V(0) = 0$, and $V(x) > 0$ for all $x \in \mathcal{X} \setminus \{0\}$.