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Handling constraints in optimal control with saturation functions and system extension

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ABSTRACT

A method is presented to systematically transform a general inequality-constrained optimal control problem (OCP) into a new equality-constrained OCP by means of saturation functions. The transformed OCP can be treated more conveniently within the standard calculus of variations compared to the original constrained OCP. In detail, state constraints are substituted by saturation functions and successively constructed dynamical subsystems, which constitute a (dynamical) system extension. The dimension of the subsystems corresponds to the relative degree (or order) of the respective state constraints. These dynamical subsystems are linked to the original dynamics via algebraic coupling equations. The approach results in a new equality-constrained OCP with extended state and input vectors. An additional regularization term is used in the cost to regularize the new OCP with respect to the new inputs. The regularization term has to be successively reduced to approach the original constrained solution. The new OCP can be solved in a convenient manner, since the stationarity conditions are easily determined and exploited. An important aspect of the saturation function formulation is that the constraints cannot be violated during the numerical solution. The approach is illustrated for an extended version of the well-known Goddard problem with thrust and dynamic pressure constraints and using a collocation method for its numerical solution.

1. Introduction

Numerical methods for the solution of optimal control problems (OCPs) can roughly be divided in two different classes. In *direct methods*, the OCP is discretized to obtain a finite-dimensional parameter optimization problem, see, e.g., [1–6]. Well-known advantages of the direct approach are the good domain of convergence as well as the efficient handling of constraints. On the other hand, *indirect approaches* are based on the calculus of variations and require the solution of a two-point boundary value problem (BVP), see, e.g., [7]. Indirect methods are known to show a fast numerical convergence in the neighborhood of the optimal solution and to deliver highly accurate solutions, which makes them particularly attractive for aerospace applications [8–12]. However, the handling of inequality constraints via Pontryagin's maximum principle [13] is in general non-trivial, since the overall structure of the BVP depends on the sequence between singular/nonsingular and unconstrained/constrained arcs (if the respective constraint is active or not) and requires a-priori knowledge of the optimal solution structure.

In order to avoid these problems in handling constraints, a method has been presented in [14] to systematically incorporate a class of state and input constraints in a new unconstrained OCP formulation, which can be solved with standard unconstrained numerics from indirect optimal control. By using the state constraints as linearizing outputs, a normal form representation of the considered nonlinear system is derived. The constraint dynamics are then substituted by means of saturation functions and successive differentiation along the normal form cascades. This concept follows an approach originally presented in the context of feedforward control design [15,16]. The procedure results in a new unconstrained system representation having the *same* system dimension but new state and input variables. However, the specific transformation and replacement technique as presented in [14] is limited to a class of state constraints with well-defined relative degree. This means, for instance, that in case of a single-input system the approach is restricted to a single state constraint.

The intention of this paper therefore is to extend the saturation function approach to a more general class of constrained OCPs. The state constraints are represented by smooth saturation functions

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whose arguments satisfy a differential equation determined by successively differentiating the state constraint function up to its relative degree, i.e. until the input appears. In this way, dynamical subsystems in new coordinates are constructed for each state constraint. These subsystems are coupled to the original dynamics via equality constraints that relate their inputs to the variables of the original system. In addition, input constraints (or mixed state-input constraints) can also be considered by direct usage of saturation functions.

The resulting system of differential-algebraic equations (DAE) with *extended* state and input vectors is used to define a new OCP with equality constraints that can be handled conveniently in the calculus of variations (or alternatively with direct methods). The necessary optimality conditions define a two-point boundary value problem, which can be solved with unconstrained numerical methods. An additional regularization term is added to the cost in order to achieve regularity of the new OCP. The corresponding regularization parameter has to be successively reduced during the numerical solution of the new OCP to approach the optimal solution of the original constrained OCP.

An intrinsic property of the saturation function approach is that the constraints cannot be violated during the numerical solution due to their inherent incorporation in the new OCP [17,14]. This is particularly advantageous for the numerical initialization.

The paper is outlined as follows: Section 2 introduces the considered class of constrained OCPs. Section 3 is devoted to the transformation of the original constrained OCP into a new equality-constrained (and regularized) OCP with extended state and input vectors. The convergence properties of the new OCP for a successively reduced regularization parameter are investigated in Section 4. Section 5 is concerned with the solution of the new OCP by deriving the optimality conditions from the calculus of variations. In addition, a collocation method is shortly introduced to numerically solve the two-point BVP stemming from the optimality conditions. Section 6 applies the method to the well-known Goddard problem with thrust and dynamic pressure constraints and discusses the numerical results. Section 7 concludes the paper and gives an outlook on potential future research activities in this field.

2. Problem statement

The following general inequality-constrained optimal control problem, called OCP_u, is considered:

minimize

$$J(u) := \varphi(x(T), T) + \int_0^T L(x, u, t)\mathrm{d}t \tag{1}$$

subject to

$$\dot{x} = f(x, u), \quad x(0) = x_0, \tag{2}$$

$$\chi(x(T), T) = 0, \tag{3}$$

$$c_i(x) \in \left[c_i^-, c_i^+\right], \quad i = 1, \ldots, p, \tag{4}$$

$$d_i(x, u) \in \left[d_i^-, d_i^+\right], \quad i = 1, \ldots, q. \tag{5}$$

It is assumed that the nonlinear system (2) with state $x \in \mathbb{R}^n$, input $u \in \mathbb{R}^m$, and $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ possesses a unique state trajectory $x(t)$ for each input trajectory $u(t)$, such that the cost (1) can be regarded as the functional $J(u)$. At the end of the time interval $t \in [0, T]$, the terminal conditions $\chi : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^l$ are imposed on the state x. The terminal time T in (1) and (3) may be fixed or unspecified. The constraints (4), (5) represent state constraints and mixed state-input constraints with interval bounds. The functions φ, L, f, χ, c_i, and d_i are assumed to be sufficiently smooth. Note that the two-sided constraints (4) and (5) are considered for the sake of generality. In practice, the constraints may also describe one-sided bounds, e.g. $c_i(x) \leq c_i^+$.

The order [7] or relative degree [18] of each state constraint function $c_i(x)$ in (4) is defined by

$$\frac{\partial c_i^{(j)}}{\partial u} = 0, \quad j = 1, \ldots, r_i - 1, \qquad \frac{\partial c_i^{(r_i)}}{\partial u} \neq 0, \tag{6}$$

with

$$\begin{aligned} c_i^{(j)}(x) &:= L_f^j c_i(x), \quad j = 1, \ldots, r_i - 1, \\ c_i^{(r_i)}(x, u) &:= L_f^{r_i} c_i(x), \quad i = 1, \ldots, p. \end{aligned} \tag{7}$$

The operator L_f denotes the Lie derivative defined by $L_f c_i(x) = \frac{\partial c_i}{\partial x} f(x, u)$ and $L_f^j c_i(x) = L_f L_f^{j-1} c_i(x)$. Literally, the order r_i corresponds to the number of times the state constraint function $c_i(x)$ has to be differentiated until at least one element of the input vector $u = (u_1, \ldots, u_m)^\mathsf{T}$ appears explicitly.[1] In addition, the mixed state-input constraints (5) are assumed to be well-defined with respect to u, i.e.

$$\frac{\partial d_i}{\partial u} \neq 0, \quad i = 1, \ldots, q. \tag{8}$$

The considered OCP_u covers a large class of optimal control problems. Note in particular that the interval bounds of the constraints (4), (5) actually represent two constraints for each function $c_i(x)$ and $d_i(x, u)$ in the standard notation of optimal control [13,7].

In the following, a method is demonstrated to compute solutions of the inequality-constrained OCP_u . Although there is no practical obstruction to using it on problems that cannot be guaranteed to possess a global solution, it simplifies the exposition to make this assumption. Therefore we formulate the following

Assumption 1. OCP_u has an optimal solution u^* (to which corresponds x^*) with the optimal cost $J(u^*) = J^*$.

3. Saturation function approach

Within the presented approach, the original inequality-constrained OCP_u is transformed into a new OCP by using saturation functions to systematically substitute the constraints (4), (5) by additional dynamical subsystems and algebraic coupling equations. The resulting DAE representation with extended state and input vectors leads to a new $\mathrm{OCP}_{\bar{u}}^{\varepsilon}$ with equality constraints. An additional regularization term with parameter ε is used to regularize $\mathrm{OCP}_{\bar{u}}^{\varepsilon}$ with respect to the introduced additional inputs.

3.1. Incorporation of state constraints

Consider in a first step the state constraints (4). The idea of the approach is to replace $c_i(x)$ by a saturation function

$$c_i(x) = \psi_i(\xi_{i,1}), \quad i = 1, \ldots, p \tag{9}$$

with the new unconstrained variable $\xi_{i,1} \in \mathbb{R}$. To represent the constraint, the variable $\xi_{i,1}$ will satisfy some well-chosen differential equation whose construction is detailed below. The saturation functions $\psi_i : \mathbb{R} \to (c_i^-, c_i^+)$ are assumed to be smooth and strictly monotonically increasing, i.e. $\mathrm{d}\psi_i/\mathrm{d}\xi_{i,1} > 0 \forall \xi_{i,1} \in \mathbb{R}$. Hence, the limits $c_i^\pm$ are only reached asymptotically for $\xi_{i,1} \to \pm\infty$, see Fig. 1.

In order to calculate the derivative $c_i^{(r_i)}(x, u)$, where the input u appears explicitly, (9) is successively differentiated and new coordinates $\xi_{i,j+1}$ are introduced for the derivatives $\dot{\xi}_{i,j} =$

[1] For the sake of clarity, the dependance of $c_i^{(r_i)}$ on u has been made explicit in (7).