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Approximate observability of infinite dimensional bilinear systems using a Volterra series expansion

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ABSTRACT

In this paper, the approximate observability of a class of infinite dimensional bilinear systems is investigated. The observability conditions are discussed based on a Volterra series representation of this class of systems. A testable observability criterion is derived for the case where the infinitesimal generator is self-adjoint or Riesz-spectral operator. The theoretical results are illustrated by examples.

1. Introduction

System theoretic properties, such as stability, controllability and observability for infinite dimensional systems have been extensively studied over the past four decades. There is a well established systems theory for infinite dimensional linear dynamical systems (Russell [1], Curtain and Zwart [2], and Zuazua [3] and the references therein). Roughly speaking, the controllability problem consists in driving the state of the system (the solution of the controlled equation under consideration) to a prescribed final target state (exactly or in some approximate way) in finite time. The observability problem concerns the problem of reconstructing the full trajectory from measured outputs. For infinite dimensional linear systems, controllability and observability form a duality relationship considering the linear system and its dual system so that these two properties can be investigated under the same framework. In this paper, we are mainly concerned about the observability of infinite dimensional bilinear systems. It is worth pointing out that observability is one of the most fundamental properties, along with controllability and stability, in control systems theory. If sensors are arranged such that a system is observable, in principle, we may uniquely reconstruct the full trajectory or the initial states of the system without detecting them. Observability indicates how one can arrange sensors to determine states with a smaller number of sensors than the number of states, and plays an important role in both finite and infinite dimensional control theory. Thus, observability is an important criterion when implementing sensors and designing state observers for physical systems.

Unlike finite-dimensional systems, there are various definitions of observability for infinite dimensional systems which in the finite dimensional case coincide. Most observability is defined in terms of the distinguishability of a pair of initial states and two important concepts are exact and approximate observability (Curtain and Zwart [2]). The definition of observability can also depend on the length of the time interval or be independent of any specific time interval (Curtain and Zwart [2]). Generally, for infinite dimensional linear systems, the control input is irrelevant to the observability whilst for nonlinear systems, observability is normally input dependent. In this paper, we concentrate on systems governed by bilinear partial differential equations and consider the approximate observability which is input dependent. There are a few results about the observability and observers of infinite dimensional bilinear systems (Belikov [4], Xu [5], Bounit and Hammouri [6], and Gauthier, Xu, and Bounabat [7]). The difference between these earlier studies and our approach is that we will use a Volterra series representation of this class of infinite dimensional bilinear systems in the analysis of observability.

The Volterra series (Volterra [8], Rugh [9]) is a functional series expansion which is generally used to represent the input/output relationship of a nonlinear system with mild nonlinearities. It has been successfully applied in the analysis of finite dimensional nonlinear control systems in the time and frequency domains (Schetzen [10], Sansen [11], Billings and Tsang [12,13], Peyton-Jones and Billings [14], Billings and Peyton-Jones [15] and the references

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therein). One of the most important problems in the Volterra series approach is the existence and convergence of such series, which has been addressed by several authors for finite dimensional nonlinear systems. d'Alessandro, Isidori, and Ruberti [16] studied the Volterra series representation for bilinear systems. Linear-analytic systems have been studied by Brockett [17] and Lesiak and Krener [18]. Other relevant theoretical results also exist (Sandberg [19], Boyd and Chua [20], Banks [21], and Peng and Lang [22]). More recently, the Volterra series expansion for infinite dimensional nonlinear systems has been investigated. Guo, et al. [23] proved the existence and convergence of a Volterra series representation for the mild solutions of a class of infinite dimensional nonlinear systems. In Hélie and Laroche [24] it was shown that a series expansion of a class of infinite dimensional bilinear systems, nonlinear in the state and affine in the input, can be obtained. Based on these previous results, in this paper, the approximate observability of a class of infinite dimensional bilinear systems is investigated using the Volterra series representation.

The paper is organized as follows. In Section 2 a formal Volterra series representation is derived for the solution of the underlying infinite dimensional bilinear systems. Section 3 gives the definition of the observability we are investigating and the observability conditions are discussed based on a Volterra series representation of this class of systems. A testable observability criterion is derived for the case where the infinitesimal generator is self-adjoint or the Riesz-spectral operator. Examples are presented in Section 4. Finally, conclusions are drawn in Section 5.

2. Preliminary

Consider the following infinite dimensional bilinear system with input u and output y

$$\dot{z}(t) = Az(t) + D(z(t), u(t)), \qquad z(0) = z_0, \quad t \geq 0$$
$$y(t) = Cz(t) \tag{1}$$

where $x(t) \in Z, u(t) \in U$, and $y(t) \in Y, Z, U$, and Y are Hilbert spaces. It is assumed that A is the infinitesimal generator of a C_0-semigroup $S(t), t \geq 0$ of bounded linear operators on Z. This means that there exist constants $M \geq 1$ and $\omega > 0$ such that

$$\|S(t)\|_{\mathcal{L}(Z)} \leq M \exp(\omega t), \quad t \geq 0. \tag{2}$$

It also implies that A is closed and densely defined in Z. Assume that C is a bounded linear operator from Z to Y, D is a bounded bilinear operator from $Z \times U$ to Z, with norm

$$\|D\|_{\mathcal{L}(Z\times U,Z)} = \sup_{\substack{z\in Z, u\in U \\ \|z\|_Z = \|u\|_U = 1}} \|D(z, u)\|_Z \tag{3}$$

and

$$\|D(z, u)\|_Z \leq L\|z\|_Z\|u\|_U \tag{4}$$

where $L > 0$ is a positive constant. In this paper, we consider the admissible controls are essentially bounded, i.e., $u \in U_a \subset L^\infty([0, \infty); U)$. Following the standard definition (e.g. Pazy [25]), for a given $T > 0$, the mild solution $z \in C([0, T]; Z)$ of (1) is defined as the solution of the following integral equation

$$z(t) = S(t)z_0 + \int_0^t S(t - \tau_1)D(z(\tau_1), u(\tau_1))d\tau_1, \quad t \in [0, T]. \tag{5}$$

The existence and uniqueness of the mild solution of system (1) over $[0, T]$ can be readily shown by using the Banach fixed point theorem with the standard arguments (e.g. Theorem 6.1.2 Pazy [25] and Theorem 2.1 Zhang and Joo [26]).

Lemma 1. *For every $z_0 \in Z$ and $u \in U_a$, the bilinear system (1) has a unique mild solution $z \in C([0, T]; Z)$, Moreover $z_0 \to z$ is Lipschitz from Z to $C([0, T]; Z)$.*

Note that the bounded bilinear operator $D : Z \times U \to Z$ induces a bounded linear operator D_1 from Z to $\mathcal{L}(U, Z)$ (the set of bounded linear operators from U to Z) as

$$D_1zu = D(z, u) \tag{6}$$

or a bounded linear operator D_2 from U to $\mathcal{L}(Z)$ (the set of bounded linear operators from Z to Z) as

$$D_2uz = D(z, u). \tag{7}$$

It follows that we can rewrite (5) as

$$\begin{aligned} z(t) &= S(t)z_0 + \int_0^t S(t - \tau_1)D_1z(\tau_1)u(\tau_1)d\tau_1 \\ &= S(t)z_0 + \int_0^t S(t - \tau_1)D_2u(\tau_1)z(\tau_1)d\tau_1. \end{aligned} \tag{8}$$

Actually, the unique mild solution of (1) can also be expressed as

$$\begin{aligned} z(t) &= S_u(t, 0)z_0 \\ &= S(t)z_0 + \int_0^t S(t - \tau_1)D_2u(\tau_1)S_u(\tau_1, 0)z_0d\tau_1 \end{aligned} \tag{9}$$

where $S_u(t, s)$ is the mild evolution operator generated by the operator $A + D_2(u), u \in U_a$ (Definition 3.2.4 and Theorem 3.2.5, Curtain and Zwart [2]).

There are several ways to derive the Volterra series representation such as the standard Picard iteration or the regular perturbation approach. A simple way to understand this is to substitute for $z(\tau)$ in (8) using an expression of this same form,

$$\begin{aligned} z(t) &= S(t)z_0 + \int_0^t S(t - \tau_1)D_1\left[S(\tau_1)z_0 \right.\\ &\quad \left. + \int_0^{\tau_1} S(\tau_1 - \tau_2)D_1z(\tau_2)u(\tau_2)d\tau_2\right]u(\tau_1)d\tau_1 \\ &= S(t)z_0 + \int_0^t S(t - \tau_1)D_1S(\tau_1)z_0u(\tau_1)d\tau_1 \\ &\quad + \int_0^t \int_0^{\tau_1} S(t - \tau_1)D_1S(\tau_1 - \tau_2)D_1z(\tau_2) \\ &\quad \times u(\tau_2)u(\tau_1)d\tau_2d\tau_1. \end{aligned} \tag{10}$$

Substituting for $z(\tau_2)$ in (10) using an expression of the form (8), and continuing in this manner yields, after $N - 1$ steps,

$$\begin{aligned} z(t) &= S(t)z_0 + \sum_{n=1}^{N-1} \int_0^t \int_0^{\tau_1} \cdots \int_0^{\tau_{n-1}} S(t - \tau_1) \\ &\quad \times D_1 \cdots D_1S(\tau_{n-1} - \tau_n)D_1S(\tau_n)z_0u(\tau_n)u(\tau_{n-1}) \\ &\quad \cdots u(\tau_1)d\tau_n \cdots d\tau_1 + \int_0^t \int_0^{\tau_1} \cdots \int_0^{\tau_{N-1}} S(t - \tau_1) \\ &\quad \times D_1S(\tau_1 - \tau_2)D_1 \cdots D_1S(\tau_{N-1} - \tau_N) \\ &\quad \times D_1z(\tau_N)u(\tau_N)u(\tau_{N-1}) \cdots u(\tau_2)u(\tau_1)d\tau_N \cdots d\tau_1 \end{aligned} \tag{11}$$

where we denote $\tau_0 = t$. The last term in (11) should approach to 0 in a uniform way on any finite time interval $[0, T]$ if the above iteration process converges. (Recall that the iteration is absolutely and uniformly convergent on $[0, T]$ in this case provided $\|u\|_{U_a} \leq \varepsilon$ and T is sufficiently small.) It follows the standard Volterra series representation of the system (1) is given by

$$\begin{aligned} z(t) &= \sum_{n=0}^{\infty} z_n(t) \\ &= h_0(t) + \sum_{n=1}^{\infty} \int_0^t \int_0^{\tau_1} \cdots \int_0^{\tau_{n-1}} h_n(t, \tau_1, \ldots, \tau_n) \\ &\quad \times u(\tau_n) \cdots u(\tau_1)d\tau_n \cdots d\tau_1 \end{aligned} \tag{12}$$