Systems & Control Letters 59 (2010) 635–638

Contents lists available at ScienceDirect

Systems & Control Letters

journal homepage: www.elsevier.com/locate/sysconle

Decay of Hankel singular values of analytic control systems

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ARTICLE INFO

Article history:
Received 1 July 2010
Received in revised form
19 July 2010
Accepted 22 July 2010
Available online 17 August 2010

Keywords:
Model order reduction
Hankel operator
Infinite-dimensional systems

ABSTRACT

We show that control systems with an analytic semigroup and control and observation operators that are not too unbounded have a Hankel operator that belongs to the Schatten class S_p for all positive p. This implies that the Hankel singular values converge to zero faster than any polynomial rate. This in turn implies fast convergence of balanced truncations. As a corollary, decay rates for the eigenvalues of the controllability and observability Gramians are also provided. Applications to the heat equation and a plate equation are given.

1. Introduction

The Hankel singular values σ_k of a transfer function $\mathsf{G} \in H^\infty$ capture how well the transfer function can be approximated by stable transfer functions of smaller McMillan degree. For all stable transfer functions G_n of McMillan degree n we have

$$\sigma_{n+1} \leq \|\mathsf{G} - \mathsf{G}_n\|_{H^\infty},$$

and there always exist stable transfer functions G_n of McMillan degree n (for example those resulting from balanced truncation [1]) such that

$$\|\mathsf{G} - \mathsf{G}_n\|_{H^\infty} \leq 2 \sum_{k=n+1}^{\infty} \sigma_k.$$

We are interested here in the case where G is irrational (and thus has infinite McMillan degree). The above estimates show that a necessary condition for convergence of finite McMillan degree approximations is compactness of the Hankel operator of G and that a sufficient condition is nuclearity (i.e. membership of the trace class) of the Hankel operator of G. These two conditions are discussed in [2]. However, the estimates provide more information: if we have a convergence rate of the Hankel singular values, then these provide a convergence rate for the (e.g. balanced truncation) approximations. For delay equations this issue is treated in detail in [3]. For discrete-time infinite-dimensional systems this is treated in [4]. For applications in random matrix theory the decay rate of Hankel singular values (under for control theory applications unreasonably strict assumptions) is studied in [5]. For finite-dimensional systems an investigation of the decay of Hankel singular values was carried out in [6], where also the decay of the eigenvalues of the systems Gramians is studied.

In this article we show that for control systems with an analytic semigroup and control and observation operators that are not too unbounded, the Hankel singular values satisfy

$$\sum_{k=1}^{\infty} \sigma_k^p < \infty,$$

(i.e. the Hankel operator is in the Schatten class S_p) for all $p > 0$. This implies that $n^q \sigma_n \to 0$ for all $q > 0$, so that the Hankel singular values of such systems decay very rapidly. The proof of this fact is based on the characterization of Schatten class Hankel operators in terms of their transfer function belonging to a certain Besov space which was proven independently by Peller [7] and Semmes [8] for the case $0 < p < 1$ most relevant here (and earlier by Peller [9,10] for the case $p \geq 1$).

In Section 2 we precisely state this characterization theorem and provide the needed definitions. In Section 3 we then state exactly which control systems we study and give some partial differential equation examples. The main result and its proof follow in Section 4. As corollaries in that section we also obtain results on the decay rate of the eigenvalues of the systems Gramians.

2. Bergman spaces, Besov spaces and the characterization theorem

The Bergman space $A^p(\mathbb{C}_0^+; \mathscr{B})$ with $p > 0$, $\mathbb{C}_0^+$ the open right half complex plane and $\mathscr{B}$ a Banach space consist of the analytic functions $f : \mathbb{C}_0^+ \to \mathscr{B}$ that satisfy

$$\int_0^\infty \int_{-\infty}^\infty \|f(x + \mathrm{i}y)\|_{\mathscr{B}}^p \, \mathrm{d}y \, \mathrm{d}x < \infty.$$

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doi:10.1016/j.sysconle.2010.07.009

The Bergman kernel (for the right half plane) is defined on $\mathbb{C}_0^+ \times \mathbb{C}_0^+$ as

$$K(z, w) := \frac{1}{(z + \bar{w})^2}. \tag{1}$$

The weighted Bergman space $A^{p,r}(\mathbb{C}_0^+; \mathscr{B})$ with $p > 0$ and $r > -\frac{1}{2}$ consists of those analytic functions $f : \mathbb{C}_0^+ \to \mathscr{B}$ that satisfy

$$\int_0^\infty \int_{-\infty}^\infty \|f(x + \mathrm{i}y)\|_{\mathscr{B}}^p K(x + \mathrm{i}y, x + \mathrm{i}y)^{-r} \, \mathrm{d}y \, \mathrm{d}x < \infty,$$

or equivalently

$$\int_0^\infty \int_{-\infty}^\infty \|f(x + \mathrm{i}y)\|_{\mathscr{B}}^p x^{2r} \, \mathrm{d}y \, \mathrm{d}x < \infty,$$

(see Semmes [8]). The Besov space $B^p(\mathbb{C}_0^+; \mathscr{B})$ consists of the analytic functions $f : \mathbb{C}_0^+ \to \mathscr{B}$ that satisfy

$$f^{\left(\frac{2}{p}\right)} \in A^p(\mathbb{C}_0^+; \mathscr{B}).$$

Here $f^{\left(\frac{2}{p}\right)}$ is a fractional derivative. Equivalently (without having to deal with fractional derivatives) the Besov space $B^p(\mathbb{C}_0^+; \mathscr{B})$ consists of the analytic functions $f : \mathbb{C}_0^+ \to \mathscr{B}$ that satisfy

$$f^{(n)} \in A^{p, \frac{np}{2} - 1}(\mathbb{C}_0^+; \mathscr{B}),$$

for some integer $n > \frac{1}{p}$ (equivalently, for all $n > \frac{1}{p}$). See e.g. Semmes [8] or Peller [7, page 490] for this definition of Besov spaces.

Remark 1. For future reference we note that with K the Bergman kernel from (1), we have for fixed $w \in \mathbb{C}_0^+$ that

$$K(\cdot, w)^\eta \in B^p(\mathbb{C}_0^+),$$

for all $\eta > 0$ and all $p > 0$. This is (up to notation) the same statement as [8, Lemma 5]. Defining for $\eta > 0$ and $w \in \mathbb{C}_0^+$ the function $f : \mathbb{C}_0^+ \to \mathbb{C}$ as

$$f(z) := K(z, w)^\eta,$$

we then see that any analytic function $g : \mathbb{C}_0^+ \to \mathscr{B}$ with the property that for all $n \in \mathbb{N}$ there exists an $M_n > 0$ such that for all $s \in \mathbb{C}_0^+$

$$\|g^{(n)}(s)\|_{\mathscr{B}} \le M_n |f^{(n)}(s)|, \tag{2}$$

belongs to $B^p(\mathbb{C}_0^+; \mathscr{B})$ for all $p > 0$.

Recall that the Schatten class S_p consists of those operators whose singular values form an $\ell^p(\mathbb{N})$ sequence. Also recall that the Hardy space $H^2(\mathbb{C}_0^+; \mathscr{H})$ with $\mathscr{H}$ a separable Hilbert space consists of the analytic functions $f : \mathbb{C}_0^+ \to \mathscr{H}$ that satisfy

$$\sup_{x>0} \int_{-\infty}^\infty \|f(x + \mathrm{i}y)\|_{\mathscr{H}}^2 \, \mathrm{d}y < \infty,$$

and carries the obvious inner-product. By taking non-tangential limits, $H^2(\mathbb{C}_0^+; \mathscr{H})$ can be identified with a closed subspace of $L^2(\mathrm{i}\mathbb{R}; \mathscr{H})$. Similarly, $H^2(\mathbb{C}_0^-; \mathscr{H})$ can be identified with the orthogonal complement of $H^2(\mathbb{C}_0^+; \mathscr{H})$ in $L^2(\mathrm{i}\mathbb{R}; \mathscr{H})$. The Hardy space $H^\infty(\mathbb{C}_0^+; \mathscr{B})$ consists of all bounded analytic functions $\mathbb{C}_0^+ \to \mathscr{B}$ and carries the obvious norm. By taking non-tangential limits, $H^\infty(\mathbb{C}_0^+; \mathscr{B})$ can be identified with a closed subspace of $L^\infty(\mathrm{i}\mathbb{R}; \mathscr{B})$. We recall that a function in $H^\infty(\mathbb{C}_0^+; \mathcal{L}(\mathscr{U}, \mathscr{Y}))$ – using the above identification – induces by multiplication a bounded operator from $L^2(\mathrm{i}\mathbb{R}; \mathscr{U})$ to $L^2(\mathrm{i}\mathbb{R}; \mathscr{Y})$. The Hankel operator associated with this $H^\infty(\mathbb{C}_0^+; \mathcal{L}(\mathscr{U}, \mathscr{Y}))$ function is obtained by restricting this multiplication operator to $H^2(\mathbb{C}_0^-; \mathscr{H})$ and projecting onto $H^2(\mathbb{C}_0^+; \mathscr{H})$ (where again the above identifications are used).

The following theorem that follows from Peller [7] and Semmes [8] characterizes Schatten class Hankel operators in terms of their transfer functions.

Theorem 2. *Let $\mathscr{U}$ and $\mathscr{Y}$ be separable Hilbert spaces at least one of which is finite-dimensional and let $p > 0$. The function $\mathsf{G} \in H^\infty(\mathbb{C}_0^+; \mathcal{L}(\mathscr{U}, \mathscr{Y}))$ has an S_p Hankel operator if and only if $\mathsf{G} \in B^p(\mathbb{C}_0^+; \mathcal{L}(\mathscr{U}, \mathscr{Y}))$.*

Proof. In Semmes [8] and Peller [7] the case where $\mathscr{U} = \mathscr{Y} = \mathbb{C}$ can be found. Peller [11, Corollary 6.9.4] includes the general case of separable Hilbert spaces $\mathscr{U}$ and $\mathscr{Y}$. This is for the disc case, but as in [7, page 490], the half-plane case can be reduced to the disc case. The condition in [11, Corollary 6.9.4] is that $\mathsf{G} \in B^p(\mathbb{C}_0^+; S_p(\mathscr{U}, \mathscr{Y}))$. This only leaves to show that $\mathsf{G} \in B^p(\mathbb{C}_0^+; \mathcal{L}(\mathscr{U}, \mathscr{Y}))$ is equivalent to $\mathsf{G} \in B^p(\mathbb{C}_0^+; S_p(\mathscr{U}, \mathscr{Y}))$ when either $\mathscr{U}$ or $\mathscr{Y}$ is finite-dimensional.

Let $n \in \mathbb{N}$ and denote the singular values of $\mathsf{G}^{(n)}(s)$ (ordered by magnitude) by $\mu_k(s)$. At most $m := \min\{\dim \mathscr{U}, \dim \mathscr{Y}\}$ of these are nonzero. We have

$$\mu_1(s)^p \le \sum_{k=1}^m \mu_k(s)^p \le m\mu_1(s)^p.$$

It follows that

$$\|\mathsf{G}^{(n)}(s)\|_{\mathcal{L}(\mathscr{U},\mathscr{Y})}^p \le \|\mathsf{G}^{(n)}(s)\|_{S_p(\mathscr{U},\mathscr{Y})}^p \le m\|\mathsf{G}^{(n)}(s)\|_{\mathcal{L}(\mathscr{U},\mathscr{Y})}^p.$$

The result then follows from the definition of Besov space. $\square$

3. Analytic control systems

We recall that associated with a strongly continuous semigroup on a Hilbert space $\mathscr{X}$ there is a scale of fractional power Hilbert spaces $\mathscr{X}_\gamma$ with $\gamma \in \mathbb{R}$. See e.g. Staffans [12, Section 3.9], Pazy [13, Section 2.6], Engel and Nagel [14, Section 2.5].

In this article we consider dynamical systems of the form

$$\dot{x}(t) = Ax(t) + Bu(t),$$
$$y(t) = Cx(t) + Du(t),$$

where A generates an exponentially stable analytic semigroup, $B \in \mathcal{L}(\mathscr{U}, \mathscr{X}_\beta)$, $C \in \mathcal{L}(\mathscr{X}_\alpha, \mathscr{Y})$, $D \in \mathcal{L}(\mathscr{U}, \mathscr{Y})$ with $\alpha - \beta < 1$ and at least one of $\mathscr{U}$ and $\mathscr{Y}$ is finite-dimensional. More information on such a system can be found in e.g. Staffans [12, Section 5.7] where among other things it is shown that the transfer function is well defined by the formula $\mathsf{G}(s) = C(sI - A)^{-1}B + D$. In Curtain–Sasane [2] it is shown that the Hankel operator of such a system (with the additional condition that *both* $\mathscr{U}$ and $\mathscr{Y}$ are finite-dimensional) is in S_1. In this article we will show – by a completely different method – that the Hankel operator of such a system is in fact in the Schatten class S_p for all $p > 0$.

We give a couple of example of such systems.

Consider the one-dimensional heat equation with Neumann (heat flux) control and Dirichlet (temperature) observation at one end and zero Dirichlet condition at the other end (to ensure that the system is exponentially stable):

$$w_t(t, \xi) = w_{\xi\xi}(t, \xi), \qquad w_\xi(t, 0) = u(t), \qquad w(t, 1) = 0,$$
$$y(t) = w(t, 0).$$

We choose as state space $\mathscr{X} = L^2(\Omega)$ with $\Omega = (0, 1)$. We then have

$$\mathscr{X}_1 = \{f \in W^{2,2}(\Omega) : f'(0) = 0, \ f(1) = 0\},$$