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Stability of optimal control of heat equation with singular potential[☆]

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In this paper, we study stability of an optimal control for a multi-dimensional heat equation with a singular potential term. A family of perturbed optimal control problems with lower power singular potentials are formulated. It is shown that when the lower powers tend to the critical power two, the optimal controls are convergent to the optimal control of the original system.

1. Introduction

The optimal control theory deals with problems of finding a control law for a given system such that a certain optimality criterion is achieved. This theory has tremendous applications in science and engineering. There are huge works attributed to optimal control for lumped parameter systems. The optimal control theory for systems governed by partial differential equations (PDEs) is one of the main research topics in distributed parameter systems control since beginning of 1960s. In the last several years, some significant progresses have been made on optimal control problems of PDEs, we refer to [1–3] and many references therein. However, the optimal control and optimal cost of a controlled system under a small perturbation have not been fully addressed. In this paper, we attempt stability (sensitivity) analysis for a multi-dimensional heat equation with a singular potential term. This heat equation is a special case of the second-order parabolic partial differential equations, which is described by

$$\begin{cases} \partial_t y(x,t) - \triangle y(x,t) - V(x) y(x,t) = \chi_\omega u(x,t) \\ \quad \text{in } \Omega \times (0,T], \\ y(x,t) = 0 \quad \text{on } \partial\Omega \times (0,T], \\ y(x,0) = y_0(x) \quad \text{in } \Omega, \end{cases} \tag{1.1}$$

where T is a positive number, $\Omega \subset \mathbb{R}^d$ $(d \geq 3)$ is a convex and bounded domain, with smooth boundary $\partial\Omega$ and $0 \in \Omega$, ω is a nonempty open domain of Ω, and χ_ω stands for the characteristic function of ω. The singular term

$$V(x) = \frac{\lambda}{|x|^2}, \qquad \lambda < \lambda_* = \frac{(d-2)^2}{4} \tag{1.2}$$

represents a potential function. The assumption (1.2) on the constant λ is crucial for the discussions in the present paper. This is because it is proved in [4] that if the initial value y_0 in the space $L^2(\Omega)$ is non-negative, then Eq. (1.1) with control $u = 0$ admits a unique global weak solution under assumption (1.2), and when $\lambda > \lambda_*$, even the local solution may not exist. For the existence and many other properties of the solutions to Eq. (1.1), we refer to [5–8], name just a few. In particular, in [8], the well-posedness of Eq. (1.1) without the sign restriction for the solution and control is thoroughly discussed from PDE's point of view. The stabilization of Eq. (1.1) is investigated in [9].

We point out that the singular potentials occur in many physical phenomena. In non-relativistic quantum mechanics, the harmonic

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oscillator and the Coulomb central potential are typical examples of such kind (see e.g., [10]). The other applications can be found in the study of near-horizon structure of black holes and dipoles.

In this paper, we are concerned with an optimal control problem of system (1.1) in the state space $L^2(\Omega)$. It is assumed that the admissible controls are taken from the following set:

$$\mathcal{U}_{ad} = \{u \in L^2(0,T;L^2(\Omega)) \mid \|u(\cdot,t)\|_{L^2(\Omega)} \le 1 \text{ for almost all } t \in [0,T]\}. \tag{1.3}$$

By classical theory, it can be easily shown that for any $y_0 \in L^2(\Omega)$, $u \in L^2(0,T;L^2(\Omega))$, there exists a unique solution $y \in C([0,T];L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$ for Eq. (1.1) under assumption (1.2) (see also Lemma 2.4 in Section 2) and we denote this solution by $y(\cdot;y_0,u)$ to represent the dependence of the solution with the control u and the initial value y_0. Throughout the paper, we use $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ to denote the usual norm and the inner product in the space $L^2(\Omega)$ respectively without specific explanation.

Let $B(0,1) \equiv \{w \in L^2(\Omega) \mid \|w\| \le 1\}$ be the closed unit ball of $L^2(\Omega)$. The optimal control problem that we are concerned with in this paper is a LQ problem with control constraint (1.3), and target state constraint $B(0,1)$ for system (1.1). The latter reads as follows:

$(\mathcal{P})$: $\inf\{J(u,y) \mid u \in \mathcal{U}_{ad}, y = y(\cdot;y_0,u)$ is the solution of (1.1) satisfying $y(T;y_0,u) \in B(0,1)\}$,

where

$$J(u,y) = \frac{1}{2}\int_0^T\int_\Omega [y^2(x,t) + u^2(x,t)]dxdt. \tag{1.4}$$

From practical implementable standpoint, the optimal control problem with state and control constraints is significant and natural. On this regard for PDEs, we refer to [11,12]. We also mention the work [13] where the nonlinear boundary control of semi-linear parabolic problems with pointwise state constraints is concerned.

The mathematical model (1.1) is the critical situation where the power of the potential term is two: $V(x) = \lambda/|x|^2$. However, this does not include all reasonable potentials. In reality, this power may vary in $(0,2)$ which is the most interested case in applications. More precisely, the potential function may take the form of

$$V_\alpha(x) = \frac{\lambda}{|x|^\alpha},$$

where the power parameter $\alpha \in (0,2)$. We remark that it is venerable physical folklore that potentials of the form $V_\alpha(x)$ product reasonable quantum dynamics as $\alpha \in (0,2)$. This is explained in details in [14, Section X.2, p. 169] and [15, Section XI.6, p. 64]), respectively. This gives rise to a family of natural optimal control problems under such potential functions as counterparts of problem $(\mathcal{P})$, which is stated as follows:

$(\mathcal{P}_\alpha)$: $\inf\{J_\alpha(u_\alpha,y_\alpha) \mid u_\alpha \in \mathcal{U}_{ad}, y_\alpha = y_\alpha(\cdot;y_0,u_\alpha)$ is the solution to Eq. (1.6) satisfying $y_\alpha(T) \in B(0,1)\}$,

where

$$J_\alpha(u_\alpha,y_\alpha) = \frac{1}{2}\int_0^T\int_\Omega [y_\alpha^2(x,t) + u_\alpha^2(x,t)]dxdt, \tag{1.5}$$

subject to control constraint $\mathcal{U}_{ad}$ defined by (1.3) and $y_\alpha = y_\alpha(\cdot;y_0,u_\alpha)$ defined by

$$\begin{cases} \partial_t y_\alpha(x,t) - \triangle y_\alpha(x,t) - V_\alpha(x)y_\alpha(x,t) = \chi_\omega u_\alpha(x,t) \\ \quad \text{in } \Omega \times (0,T], \\ y_\alpha(x,t) = 0 \quad \text{on } \partial\Omega \times (0,T], \\ y_\alpha(x,0) = y_0(x) \quad \text{in } \Omega. \end{cases} \tag{1.6}$$

It is seen that system (1.6) is just system (1.1) by simply replacing V_α with V.

Same as Eq. (1.1), under assumption (1.2), for any $y_0 \in L^2(\Omega)$, $u_\alpha \in L^2(0,T;L^2(\Omega))$, it can be shown that for any $\alpha \in [2-\varepsilon, 2)$ with sufficiently small $\varepsilon > 0$, there exists a unique (weak) solution $y_\alpha \in C([0,T];L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$ to Eq. (1.6). This is discussed in Lemma 2.4 in detail in the next section.

Clearly, problem $(\mathcal{P}_\alpha)$ can be regarded as a perturbed problem of problem $(\mathcal{P})$ with the perturbed operator:

$$V_{er}(x) = \left(\frac{\lambda}{|x|^\alpha} - \frac{\lambda}{|x|^2}\right)I, \quad \alpha \in (0,2) \tag{1.7}$$

where I is the identity operator in $L^2(\Omega)$. By the Hardy–Poincaré inequality, this perturbed operator is not a bounded operator in $L^2(\Omega)$ for any $\alpha \in [2-\varepsilon,2)$. This fact measures the degree of major difficulty of the problem and surmounting the obstacle is the main contribution of the present work. Our objective is to relate the optimal controls between the problem $(\mathcal{P})$ and the perturbed problem $(\mathcal{P}_\alpha)$. To this purpose, we must first ensure the existence of feasible pairs for problems $(\mathcal{P})$ and $(\mathcal{P}_\alpha)$, respectively. The following assumption is made for problem $(\mathcal{P})$ throughout the paper.

Assumption (S). For given $y_0 \in L^2(\Omega)$, there exists an admissible control $u_0 \in \mathcal{U}_{ad}$ such that the corresponding solution $y(\cdot;y_0,u_0)$ to Eq. (1.1) reaches the interior of $B(0,1)$: $y(T;y_0,u_0) \in \text{int}(B(0,1))$.

The Assumption (S) is called the Slater condition (see, e.g., [12]) which guarantees the existence of feasible pairs for problem $(\mathcal{P})$ and $(\mathcal{P}_\alpha)$ for all $\alpha \in [2-\varepsilon,2)$ with sufficiently small $\varepsilon > 0$. The latter is explained in the next section. We point out that the Assumption (S) is reasonable in the sense that Assumption (S) is always true for any sufficient large $T > 0$ or for any fixed $T > 0$ with sufficiently small initial norm $\|y_0\|_{L^2(\Omega)}$, which is explained in Remark 2.2 in next section.

Since both $J(\cdot,\cdot)$ and $J_\alpha(\cdot,\cdot)$ are strictly convex functionals, it is easily shown that under Assumption (S), the optimal control problems $(\mathcal{P})$ and $(\mathcal{P}_\alpha)$ for all $\alpha \in [2-\varepsilon,2)$ with sufficiently small $\varepsilon > 0$ admit unique solutions which are denoted by (u^*,y^*) and (u_α^*,y_α^*), respectively. We refer this conclusion to Theorem 1.1 of [16]. The main result of this paper is the following Theorem 1.1.

Theorem 1.1. *Let $y_0 \in L^2(\Omega)$ and assume that Condition (S) stands. Suppose that (u^*,y^*) is the optimal pair for problem $(\mathcal{P})$ and (u_α^*,y_α^*) is the optimal pair for problem $(\mathcal{P}_\alpha)$. Then*

$$u_\alpha^* \to u^* \quad \text{strongly in } L^2(0,T;L^2(\Omega)) \text{ as } \alpha \uparrow 2, \tag{1.8}$$

and

$$J_\alpha(u_\alpha^*,y_\alpha^*) \to J(u^*,y^*) \quad \text{as } \alpha \uparrow 2. \tag{1.9}$$

In addition,

$$y_\alpha^*(T;y_0,u_\alpha^*) \to y^*(T;y_0,u^*) \quad \text{strongly in } L^2(\Omega) \text{ as } \alpha \uparrow 2. \tag{1.10}$$

It is apparently that Theorem 1.1 can be regarded as a sensitivity or stability for the optimal control pair and optimal cost for problem $(\mathcal{P})$. Mathematically, the major difficulty in proving Theorem 1.1 lies in the regularity of the solution caused by the singular potential terms. Simply speaking, we cannot expect $H^2(\Omega)$ regularity in spatial variable either for solution of Eq. (1.1) or (1.6) because the perturbed operator V_{er} defined by (1.7) is not a bounded operator in $L^2(\Omega)$ as $\alpha \uparrow 2$. This gives rise to the difficulty in application of the classical perturbation theory of C_0-semigroups (see, e.g., [17]). This difficulty is surmounted, however, by setting up new function space in terms of the Hardy–Poincaré inequality, which enables us to improve the regularity in the space $H^{-1}(\Omega)$.

We proceed as follows. In Section 2, we give some preliminary results. Section 3 is devoted to the proof of Theorem 1.1. Some concluding remarks are presented in Section 4.