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Coprime factor model reduction for discrete-time uncertain systems

Li Li *

School of Electrical, Mechanical and Mechatronic Systems, University of Technology, Sydney, Broadway NSW 2007, Australia

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ABSTRACT

This paper presents a contractive coprime factor model reduction approach for discrete-time uncertain systems of LFT form with norm bounded structured uncertainty. A systematic approach is proposed for coprime factorization and contractive coprime factorization of the underlying uncertain systems. The proposed coprime factor approach overcomes the robust stability restriction on the underlying systems which is required in the balanced truncation approach. Our method is based on the use of LMIs to construct the desired reduced dimension uncertain system model. Closed-loop robustness is discussed under additive coprime factor perturbations.

1. Introduction

This paper addresses the coprime factorization (CF) and model reduction problems for discrete-time uncertain systems which are possibly *robustly unstable*. The uncertain systems under consideration are described in terms of linear fractional transformations (LFTs) [1] with structured norm bounded uncertainty.

Model reduction has been an active research area in the control society since 1960s. A large number of model reduction methods have appeared in the literature, among which one of the most commonly applied methods for stable linear time invariant (LTI) systems is the balanced truncation method [2] with guaranteed error bounds [3,4]. For unstable LTI systems, a coprime factor approach [5] is proposed to avoid the stability issues. Discrete-time related topics can be found, for example, in [6,7] and the references therein.

Model reduction problems for uncertain systems have attracted much attention in recent years; see, for example, LFT systems [8–13], gain scheduling [14,15], linear parameter-varying systems [16–19], linear time-varying systems [20,21], nonlinear systems [22], linear parameter dependent (LPD) systems [23], and related approximation, truncation and simplification problems [24,25]. The balanced truncation method for *robustly stable* uncertain systems is studied in [8,9,26] within the LFT framework. Concerning those uncertain systems which may be *robustly unstable*, a coprime factorization based approach is proposed in [12], which extends coprime factor approach [5] for LTI systems to the underlying uncertain systems. However, no indication is given in [12] on the contractiveness of the resulting coprime factors. This motivates the question as to whether a contractive CF can be obtained for uncertain systems. Contractive CF, as an alternative to normalized CF, has properties similar to normalized CF. In the meanwhile, it enables us to take advantage of linear matrix inequality (LMI) techniques, providing more flexibility to accommodate structure constraints including topological structures and uncertainty structures, and thus can be effectively solved by available softwares. Particularly for discrete-time uncertain systems, contractive CF is motivated by the following two observations. Firstly, for discrete-time LTI systems, applying balanced truncation to normalized coprime factors of original systems would result in contractive coprime factors of reduced systems, rather than normalized ones as in continuous cases. Therefore, it is not necessary to consider normalized CF in the first place in balanced truncation approaches. Secondly, in the presence of uncertainty, it is very difficult to obtain normalized coprime factors for the underlying systems because the corresponding Riccati equations are hard to solve and most probably lead to infeasible solutions.

In this paper, the coprime factor model reduction problem studied in [12] is revisited. The study of this problem is based on the results in [12] and the author's previous work on uncertain systems [26,10,13,23]. In [26,10,23], model reduction problems for two classes of continuous uncertain systems are studied. By introducing generalized controllability and observability Gramians, balanced truncation and balanced LQG truncation model reduction approaches are investigated. In this paper, following the idea of balanced LQG truncation, instead of just balancing the solutions

* Tel.: +61 295149961.
E-mail address: Li.Li@uts.edu.au.

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to the control/filter Riccati inequalities, coprime factors are constructed based on Riccati inequalities. The advantage of coprime factor model reduction over balanced LQG truncation is that coprime factor model reduction can provide quantitative robust stability margin which will be discussed in Section 5. In [13], coprime factor model reduction for a class of continuous-time uncertain systems is investigated. In this paper the method is extended to discrete uncertain systems. It should be emphasized here that this extension is not trivial because of the significant difference between continuous and discrete systems especially in the presence of uncertainty. The contribution of this paper, compared to the results of [12], is three fold. Firstly, we eliminate the full column rank restriction on the B-matrix in [12], providing a more general solution to constructing coprime factorization for uncertain systems. Secondly, a systematic approach to obtain the coprime factorization for the underlying uncertain systems is presented based on the use of LMIs. A sufficient and necessary condition to ensure the feasibility of the derived LMI is also specified. Contractiveness is subsequently accomplished by choosing a specific feedback gain, which extends the similar LTI results to the uncertain systems under consideration. This enables us to apply balanced truncation [8,9] to the resulting contractive coprime factors to obtain the reduced-order uncertain systems. It is shown that the resulting reduced coprime factors are contractive as well. Thirdly, closed-loop robustness is discussed for the reduced uncertain system under model reduction error on coprime factors. A sufficient condition is presented to guarantee the closed-loop stability when the original model is replaced by the reduced model. This robustness property could potentially contribute to the analysis of gap metric for uncertain systems which will be the topic of future research. Although in this paper we only focus on the uncertain systems, the results can be readily applied to multidimensional systems by replacing the uncertainty variables with frequency parameters. A preliminary version of this work appeared in [27].

While this paper focuses on model reduction problems, its results related to coprime factorization of uncertain systems could be used in many branches of robust control problems, for example, analysis of gap metric [28] for uncertain systems, robust controller design using coprime factorization [29] and Youla parametrization [30], to list a few. Future work will be carried on to apply the results of this paper to other relevant control problems.

Notation. The notation is quite standard. $\mathbf{R}^{m\times n}$ and $\mathbf{C}^{m\times n}$ denote the set of real and complex, $m \times n$ matrices, and $\mathbf{H}^m$ denotes the set of Hermitian $m \times m$ matrices. Let l^m and l_2^m be the space of all the sequences and square summable sequences in $\mathbf{R}^m$ respectively. Let $\mathcal{L}(l^m)$ denote the space of all linear operators mapping from l^m to l^m, and $\mathcal{L}(l_2^m)$ denote the space of all linear bounded operators mapping from l_2^m to l_2^m. The gain of an operator Δ in $\mathcal{L}(l_2^m)$ is given by $\|\Delta\| = \sup_{z\in l_2^m, z\neq 0} \frac{\|\Delta z\|}{\|z\|}$, and the adjoint operator of Δ is denoted as Δ^* if Δ is linear, and if $\Delta = \Delta^*$, $\Delta < 0$ means that $x^*\Delta x < 0$ for any $x \neq 0$ in $\mathbf{R}^m$. We also use M^* to denote the complex conjugate transpose of a complex matrix M. $FM(\cdot)^*$ and $(\cdot)^*MF$ denote FMF^* and F^*MF respectively for a Hermitian matrix M.

2. Problem formulation

We consider the uncertainty structure

$$\mathbf{\Delta^c} = \{\mathrm{diag}(\delta_1 I_{h_1}, \ldots, \delta_k I_{h_k}) : \delta_i \in \mathcal{L}(l_2), \delta_i \text{ causal}, \|\delta_i\| \le 1\},$$

and the following uncertain system:

$$\mathcal{G}_\Delta : \begin{cases} \begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \xi \\ u \end{bmatrix}, \\ \xi = \Delta z, \quad \Delta \in \mathbf{\Delta^c}, \end{cases} \tag{1}$$

where $u(t) \in \mathbf{R}^m$ is the *control input*, $z(t) \in \mathbf{R}^h$ is the *uncertainty output*, $y(t) \in \mathbf{R}^l$ is the *measured output* and $\xi(t) \in \mathbf{R}^h$ is the *uncertainty input*; here $h = h_1 + \cdots + h_k$. Similar to the typical setting for one-dimensional discrete-time uncertain systems, we define $\delta_1 = z^{-1}$, the time shift operator, and other δ_i's are regarded as uncertainties.

Let the nominal system be denoted by $G = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$. Then, the uncertain system (1) is defined by an LFT representation as follows. For any bounded linear operator $\Delta \in \mathcal{L}(l_2^h)$ such that $I - A\Delta$ is nonsingular, define $\mathcal{F}_u(G, \Delta) := D + C\Delta(I - A\Delta)^{-1}B$. In what follows, robust stability, stabilizability and detectability of the uncertain system (1) are defined.

Definition 1 (*Robust Stability [12]*). The uncertain system (1) is said to be *robustly stable*, or equivalently, $(A, \mathbf{\Delta^c})$ is said to be *robustly stable*, if $(I - A\Delta)^{-1}$ exists in $\mathcal{L}(l_2^h)$ and is causal, for all $\Delta \in \mathbf{\Delta^c}$.

Definition 2. The uncertain system (1) is said to be *robustly stabilizable* if there exists a matrix F, such that $(A + BF, \mathbf{\Delta^c})$ is robustly stable. Similarly, the system (1) is said to be *robustly detectable* if the dual of the system (1) is robustly stabilizable.

The following lemma from [12] states a necessary and sufficient condition for robust stability. This lemma is given in terms of the positive commutant set corresponding to $\mathbf{\Delta^c}$ defined as

$$\boldsymbol{P_\Theta} = \{\mathrm{diag}(\Theta_1, \ldots, \Theta_k) : \Theta_i \in \mathbf{H}^{h_i}, \Theta_i > 0\}. \tag{2}$$

Lemma 3 (*See [12, Proposition 3 and Remark 4]*). *The system (1) is robustly stable if and only if there exists $P \in \boldsymbol{P_\Theta}$, such that*

$$APA^* - P < 0. \tag{3}$$

3. Balanced truncation

In this section we briefly review the balanced truncation model reduction technique for the uncertain system (1) presented in [8,9]. It is assumed in this section that the uncertain system (1) is robustly stable. Similar to the LTI balanced truncation approach [2–4], this robust stability assumption is essential for the balanced truncation of the uncertain system (1), and guarantees the existence of the solutions $S, P \in \boldsymbol{P_\Theta}$ to following Lyapunov inequalities,

$$ASA^* - S + BB^* < 0, \tag{4}$$

$$A^*PA - P + C^*C < 0. \tag{5}$$

Theorem 4 (*[12, Remark 4]*). *The following statements are equivalent:*

(i) *The uncertain system (1) is robustly stable.*
(ii) *The LMI (4) admits a solution $S \in \boldsymbol{P_\Theta}$.*
(iii) *The LMI (5) admits a solution $P \in \boldsymbol{P_\Theta}$.*

Definition 5. An uncertain system of the form (1) is said to be *balanced* if it has solutions to (4) and (5) which are identical diagonal matrices.

We summarize the proposed model reduction algorithm as follows.

Procedure 6 (*Balanced Truncation*).

1. Solve the LMIs (4) and (5) to obtain $S = \mathrm{diag}(S_1, \ldots, S_k) \in \boldsymbol{P_\Theta}$, $P = \mathrm{diag}(P_1, \ldots, P_k) \in \boldsymbol{P_\Theta}$.