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A Volterra series representation for a class of nonlinear infinite dimensional systems with periodic boundary conditions

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ABSTRACT

This paper proves the existence of a Volterra series representation for the mild solutions of a class of nonlinear infinite dimensional systems. More specifically, given the evolutionary system/operator $\{U(t,s) : 0 \leq s \leq t < \infty\}$ associated with a semilinear evolution equation $\partial u/\partial t = \partial^2 u/\partial x^2 + f(u)$, $u(0) = u_0 \in X$ with periodic boundary conditions, it is proved that, under suitable conditions, the unique (mild) solution $u(t) = U(t,0)u(0)$, $t \geq 0$ can be expanded by a Volterra series. A recursive algorithm is given to construct the Volterra kernels/series terms and a nonlinear heat equation is discussed to illustrate the proposed method.

1. Introduction

In the last three decades, infinite dimensional linear system theory has been extensively studied under the framework of a theory of strongly continuous semigroups on certain appropriate Banach spaces. The basic concepts in finite-dimensional linear dynamical system theory, such as transfer functions, state space representation, controllability, observability, and stability, have been successfully extended to the study of infinite dimensional linear systems ([1,2] and references therein). In recent years, an increasing interest has been observed in the study of semigroup theory of nonlinear operators and system theory of infinite dimensional nonlinear systems, which have found many applications in the theory of partial differential equations (PDEs) [3,4]. Under the semigroup theory framework, an important problem is to investigate the existence and behaviour of a strongly continuous semigroup $T(t)$ or an evolution system $\{U(t,s) : 0 \leq s \leq t < \infty\}$ associated with an initial value problem for linear or nonlinear (particularly semilinear) evolution equations. These studies reveal the relationships between semigroups (evolution systems), infinitesimal generators, and (mild) solutions to the given initial value problems [4,5].

In this paper, we are going to investigate the initial value problem of semilinear evolution equations with periodic boundary conditions by using a Volterra series method. The first motivation for this study is the successful application of Volterra series in dealing with nonlinear finite dimensional dynamic systems, particularly the development of nonlinear frequency domain theory ([6–10] and the references therein). This Volterra series based methodology has found many successful applications such as in analysing harmonic distortions and intermodulation distortions in circuits [11].

The second motivation of the paper is that recently Volterra series have been successfully applied in the analysis and synthesis of infinite dimensional dynamical systems. Vazquez and Krsric [12,13] have proposed a boundary control approach to one-dimensional parabolic PDEs whose nonlinearities are expressed as Volterra series. Based on this Volterra series model, a controller, which is in a form of Volterra series, was developed to stabilise the systems. Another application is the sound synthesis of a nonlinear string proposed by Helie and Roze [14], Helie and Hasler [15] where the nonlinear relationship between output (the displacement of the string) and input (the excitation forces) was assumed to be given by a Volterra series. It has been shown that an effective synthesis can be carried out based on this Volterra representation and decomposing the Volterra kernels on the modal basis revealed the nonlinear dynamics of each spatial modes precisely. These encouraging results show the great potential and power that the Volterra series could have in system and control community. However, the existence of such Volterra series representation for nonlinear infinite dimensional dynamical systems is still an open problem. The existence and convergence problem of Volterra series expansion for finite dimensional systems have been extensively studied through 1970s and 1980s. d'Alessandro et al. [16] studied the Volterra series expansion for bilinear systems in 1974. The convergence problem of Volterra series was then examined by Gilbert [17] and Brockett [18], where Brockett [18] showed that linear analytic systems admit a Volterra series expansion provided there is no finite escape time. Lesiak and Krener[19] and Sandberg [20] extended these existence and uniqueness results to more general systems.

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In this paper, we will prove the existence of a Volterra series representation for the mild solutions of a class of nonlinear infinite dimensional systems, in particular, semilinear evolution equations with periodic boundary conditions. More specifically, we are seeking a Volterra series representation to the (mild) solution $u(t) = U(t,s)u(s), t \geq s$ to this class of semilinear evolution equations, where $\{U(t,s) : 0 \leq s \leq t < \infty\}$ are the evolutionary system associated with this initial value problem. To the best of the authors' knowledge, this is the first result of this kind in the study of nonlinear infinite dimensional dynamical systems in the sense that the Volterra series is with respect to spatio-temporal domain rather than a purely temporal domain. Note that although we are discussing the evolution of systems with initial conditions, this type of systems certainly can be considered as closed-loop systems with certain type of feedbacks.

We start the investigation with an introduction to the problem of the existence of the solutions of this class of semilinear evolution equations [4] in Section 2. In Section 3 a theorem is given for the existence of the Volterra series for the solutions of the underlying nonlinear evolution equation $\partial u/\partial t = \partial^2 u/\partial x^2 + f(u), u(0) = u_0 \in X$ with periodic boundary conditions, X is a Banach space. The basic idea behind this is that the nonlinearities in the nonlinear evolution equations can be expanded as a convergent power series within some neighbourhood of its equilibrium if it is analytic. Then by successively replacing the nonlinearities with their power series expansions with respect to the (mild) solution of the equation the desired Volterra series representation can then be obtained. In Section 4 an iterative algorithm is given to construct the Volterra kernels and a nonlinear heat equation is discussed to illustrate the proposed theory and method. Conclusions are drawn in Section 5.

2. Preliminaries

Consider the following semilinear initial value problem in R with a periodic boundary condition

$$\frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t) + f(u(x,t)), \quad 0 < x < l, t > 0$$
$$u(0,t) = u(l,t), \qquad \frac{\partial u(0,t)}{\partial x} = \frac{\partial u(l,t)}{\partial x}, \quad t \geq 0$$
$$u(x,0) = u_0(x) \tag{1}$$

where $u(x,t) \in R$ is the state variable of the system with the spatial variable $x \in [0,l]$ and $t > 0$ denotes time. $f : R \to R$ is a real valued nonlinear function. The objective of this paper is to investigate the existence of a Volterra series expansion to the solution of the initial value problem (1). First, we will recall some basic results about the existence and uniqueness of the (mild) solution to the problem (1).

Let $X = C_p([0,l]) = \{u \in C([0,l]) | u(0) = u(l)\}$ be the Banach space of all continuous real valued periodic functions over the interval $[0,l]$ with the supremum norm $\|u\| = \sup_{x\in[0,l]} |u(x)|$. Let $A = \partial^2 u/\partial x^2$ be the linear operator in X defined by $D(A) = \{u \in X | \partial u/\partial x, \partial^2 u/\partial x^2 \in X\}$. Note that the function f induces another function $F : X \to X$ by defining

$$F(u(t))(x) = f(u(x,t)) \tag{2}$$

for all $x \in [0,l]$ and $t > 0$. The following lemmas are all from section 8.2, Pazy [4], which are listed here for completeness.

Lemma 1 (*Pazy [4], Lemma 2.1, p. 234*). *The operator A defined above is the infinitesimal generator of a compact analytic semigroup $T(t), t \geq 0$ on X.*

Lemma 2 (*Pazy [4], Theorem 2.2, p. 235*). *For every continuous real valued function f and every $u_0 \in X$ there exists a $t_0 > 0$ such that the initial value problem (1) has a unique mild solution $u(x,t)$ on $[0,t_0)$ and either $t_0 = \infty$ or if $t_0 < \infty$ then $\limsup_{t\to t_0} \|u(t)\| = \infty$. The mild solution is given by*

$$u(t) = T(t) \cdot u_0 + \int_0^t T(t-\tau) \cdot F(u(\tau)) d\tau \tag{3}$$

where the integral is in the Bochner sense and F is given by (2).

A function $f : R \to R$ is Hölder continuous with exponent $\theta, 0 < \theta < 1$ on R if there is a constant L such that

$$|f(t) - f(s)| \leq L|t-s|^\theta, \quad s,t \in I. \tag{4}$$

Lemma 3 (*Pazy [4], Theorem 2.3, p. 235*). *If the real valued function f is Hölder continuous then for every $u_0 \in X$ there exists a $t_0 > 0$ such that the initial value problem (1) has a unique classical solution $u(x,t)$ on $[0,t_0)$ and either $t_0 = \infty$ or if $t_0 < \infty$ then $\limsup_{t\to t_0} \|u(t)\| = \infty$.*

Lemma 4 (*Pazy [4], Theorem 2.7, p. 237*). *If the real valued function f is Hölder continuous and $sf(s) < 0$ for all $s \neq 0$, $s \in R$ then all solutions of the initial value problem (1) are bounded and moreover, all solutions of (1) tends to zero as $t \to \infty$.*

Note that Eq. (1) is said to generate an evolutionary system $\{U(t,s) : 0 \leq s \leq t < \infty\}$ if for every $u_0 = u(s) \in X$ the function $U(t,s)u_0, t \geq s$ is the unique solution of integral equation (3) [5]. For autonomous systems, we simply write $U(t,s) = U(t-s)$. From Lemma 4, the global mild solution exists on the interval $[0,l] \times [0,\infty)$. We will discuss the existence of a Volterra series representation over any closed sub-interval $[0,l] \times [0,T], T < \infty$. The following definition gives the Volterra series representation of a (mild) solution $u(t) = U(t)u_0$ on $[0,l] \times [0,T], T < \infty$.

Definition 1. Let $u(t) = U(t)u_0$ be the unique (mild) solution of (1) on $[0,T], T < \infty$, the operator $U(t) : X \to X$ has a Volterra series representation if there exists a set of piecewise continuous kernel functions $h_n : H_n \to R^1, n = 1,2,\ldots$, where $H_n = \{(x,t;\xi_1,\xi_2,\ldots,\xi_n) | 0 \leq x, \xi_i \leq l, t \in [0,T], i = 1,\ldots,n\}$ such that there exists a $\delta > 0$ such that whenever $\|u_0\| < \delta$,

$$U(t)u_0(x) = \sum_{n=1}^{\infty} \int_0^l \cdots \int_0^l h_n(x,t;\xi_1,\ldots,\xi_n) \times u_0(\xi_1)\cdots u_0(\xi_n) d\xi_1 \cdots d\xi_n. \tag{5}$$

The series converges absolutely and uniformly on $[0,l] \times [0,T]$ for all $\|u_0\| < \delta$.

3. The existence of a Volterra series expansion

In this section it will be shown that, under suitable conditions on the nonlinear function f, there exists a Volterra series expansion for the mild solution (3). Note that any classical or strong solution of (1) satisfies the integral equation (3). Now, consider the case where the nonlinear function $f : R \to R$ is analytic near 0. Here we make it clear that this analyticity admits two interpretations. It can be viewed as a real-value function $f : R \to R$: $f(s) = \sum_{m=2}^{\infty} a_m s^m$ within some neighbourhood of $0 \in R$ or a function $F : X \to X$: $F(u)(x) = f(u(x)) = \sum_{m=2}^{\infty} a_m u(x)^m$ within some neighbourhood of $0 \in X$. The following result is a direct consequence of the analyticity and Lemmas 1, 2 and 4.