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Characterization of linear differential systems (in several variables)

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ABSTRACT

We show that a set of smooth trajectories is the solution set of a linear constant coefficient partial differential equation if and only if it is linear, shift-invariant and complete. (By completeness, we mean exactly what Willems called jet-completeness in his Automatica paper in 1986.)

1. Introduction

In all that follows n is the number of coordinates. We shall write ∂_p $(p = 1, \ldots, n)$ to denote the partial differentiation operators acting on $C^\infty(\mathbb{R}^n)$. Recall that a multi-index is an n-tuple of nonnegative integers, i.e., an element of $\mathbb{Z}_+^n$. For a multi-index $i = (i_1, \ldots, i_n)$, it is usual to write ∂^i for $\partial_1^{i_1} \ldots \partial_n^{i_n}$. If k is a nonnegative integer, we let $\Delta(k)$ denote the set of multi-indices of order less than or equal to k. (The order of $i = (i_1, \ldots, i_n)$ is defined to be $|i| = i_1 + \cdots + i_n$.)

Given a trajectory $w \in C^\infty(\mathbb{R}^n)$, a time $t \in \mathbb{R}^n$ and a nonnegative integer $k \in \mathbb{Z}_+$, one defines the k-jet $J_t^k(w)$ of w at t as follows:

$$J_t^k(w)(i) = \partial^i w(t), i \in \Delta(k).$$

For t and k as above, we thus have a mapping

$$J_t^k : C^\infty(\mathbb{R}^n) \to \mathbb{R}^{\Delta(k)}.$$

Let $\mathcal{B}$ be a subset of $C^\infty(\mathbb{R}^n)^q$, a (continuous-time) dynamical system with the signal number q. For every time $t \in \mathbb{R}^n$ and every nonnegative integer $k \in \mathbb{Z}_+$, we let $\mathcal{B}|_{t,k}$ denote the image of $\mathcal{B}$ under

$$J_t^k : C^\infty(\mathbb{R}^n)^q \to (\mathbb{R}^{\Delta(k)})^q.$$

The sets $\mathcal{B}|_{t,k}$ seem to be interesting invariants, and a natural idea, due to Willems [1], is to try to get information about the trajectories of $\mathcal{B}$ looking at all of them. This idea leads to what Willems called jet-completeness (see Section 6 in Willems [1]), but that we just call *completeness* here. The definition of this important notion runs like this. The dynamical system $\mathcal{B}$ is complete if it satisfies the following condition

$$\forall w \in C^\infty(\mathbb{R}^n)^q, \quad (w \in \mathcal{B} \Leftrightarrow J_t^k(w) \in \mathcal{B}|_{t,k} \quad \forall t, k).$$

By definition, completeness of $\mathcal{B}$ means that the jet-spaces $\mathcal{B}|_{t,k}$ contain complete information about the trajectories of $\mathcal{B}$.

In this article, we are going to show that the solution sets of linear constant coefficient partial differential equations are exactly those dynamical systems that are linear, shift-invariant and complete.

It should be interesting to compare the result with the existing discrete-time counterpart due to Willems. Let $C(\mathbb{Z}_+^n)$ be the space of all real-valued functions on $\mathbb{Z}_+^n$, and let $\sigma_p (p = 1, \ldots, n)$ be the partial shift operators in it. We remind that a dynamical system $\mathcal{B} \subseteq C(\mathbb{Z}_+^n)^q$ is said to be complete if

$$\forall w \in C(\mathbb{Z}_+^n)^q, \quad (w \in \mathcal{B} \Leftrightarrow w|_{\Delta(k)} \in \mathcal{B}|_{\Delta(k)} \forall k).$$

The Willems theorem states that $\mathcal{B}$ is the solution set of a linear constant coefficient partial difference equation if and only if it is linear, shift-invariant and complete. (Willems [1] proved the theorem for dimension 1, and it was extended then to higher dimensions by Oberst [2] and Rocha [3]; however, this extension is easy.) Notice that, for $w \in C(\mathbb{Z}_+^n)$ and $i \in \mathbb{Z}_+^n$, we have:

$$w(i) = \sigma^i w(0).$$

(Here $\sigma^i = \sigma_1^{i_1} \ldots \sigma_n^{i_n}$, where $i_1, \ldots, i_n$ are the components of i.) In view of this, $w|_{\Delta(k)}$ can be interpreted as the k-jet of w at 0. We therefore can regard the truncated spaces $\mathcal{B}|_{\Delta(k)}$ as the jet-spaces of $\mathcal{B}$ at 0. Thus, in the definition of completeness jet-spaces at 0 only are involved.

The point of the continuous-time case is that in $C^\infty(\mathbb{R}^n)$ there are a lot of flat functions. (A function $w \in C^\infty(\mathbb{R}^n)$ is said to be flat at time t, if all its derivatives $\partial^i w$ vanish at t.) A priori is clear that if

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$\mathcal{B} \subseteq C^\infty(\mathbb{R}^n)^q$, then the trajectories of $\mathcal{B}$ that are flat at time 0 by no means can be recovered from the knowledge of the jet-spaces $\mathcal{B}|_{0,k}$. For this reason, in order to define the completeness property, it is necessary to bring into play jet-spaces at *all times*.

We equip $C(\mathbb{Z}_+^n) = \mathbb{R}^{\mathbb{Z}_+^n}$ with the product topology. It is worth noting that this topology coincides with the pointwise convergence topology. (On the field of real numbers we consider the ordinary topology defined by the absolute value $|\ |$.)

We let $s_1, \ldots, s_n$ be indeterminates. We put $s = (s_1, \ldots, s_n)$ and write $\mathbb{R}[s]$ for the ring of polynomials in $s_1, \ldots, s_n$. Likewise, we put

$$\partial = (\partial_1, \ldots, \partial_n) \quad \text{and} \quad \sigma = (\sigma_1, \ldots, \sigma_n).$$

We let $A(\mathbb{Z}_+^n)$ denote the set of all $a \in C(\mathbb{Z}_+^n)$ such that the power series (in $\mathbb{R}^n$)

$$L(a)(x) = \sum_{i \in \mathbb{Z}_+^n} a(i) \frac{x^i}{i!}$$

is uniformly convergent on compact subsets of $\mathbb{R}^n$. (Here $x^i = x_1^{i_1} \cdots x_r^{i_n}$ and $i! = i_1! \cdots i_n!$.) By the very definition, the functions $L(a)$ are entire analytic functions. It is clear that $A(\mathbb{Z}_+^n)$ is invariant with respect to the partial shift operators. (Hence these operators make it a module over $\mathbb{R}[s]$.)

We shall use "tr" for the transpose. For any topological vector space V, V^* will denote the space of continuous linear functionals on V. For any integer $k \geq 0$, $\mathbb{R}[s]_{\leq k}$ will denote the set of polynomials of degree $\leq k$.

The prerequisite for this article is Oberst's theorem (see Oberst [4]), which is a basic fact and which says that $A(\mathbb{Z}_+^n)$ is a cogenerator module. We remind that a module $\mathcal{U}$ is called a cogenerator module if for every module M and every $0 \neq x \in M$, there exists a homomorphism $\phi : M \to \mathcal{U}$ such that $\phi(x) \neq 0$. (A proof of Oberst's theorem can be found also in [5].) We shall make use of also the well-known Hahn–Banach theorem in the following formulation. If X is a subspace of a Hausdorff locally convex space V, then an element $v \in V$ belongs to the closure $\overline{X}$ of X if and only if there is no continuous linear functional f on V such that $f|_X = 0$ but $f(v) \neq 0$. (See Theorem 3.5 in [6].)

Throughout, q is a fixed positive integer number.

2. Preliminaries

Consider the canonical bilinear form

$$\mathbb{R}[s]^q \times C(\mathbb{Z}_+^n)^q \to \mathbb{R},$$

given by

$$\langle f, g \rangle = (f^{tr}(\sigma)g)(0).$$

This bilinear form is important as it permits us to identify the space of continuous linear functionals on $C(\mathbb{Z}_+^n)^q$ with $\mathbb{R}[s]^q$. Let "$\perp$" denote the orthogonal complement with respect to this bilinear form.

For every $u \in \mathbb{R}[s]^l$ and $g \in C(\mathbb{Z}_+^n)^q$, we have

$$\langle R^{tr}u, g \rangle = (u^{tr}(\sigma)R(\sigma)g)(0). \tag{1}$$

Using this formula, it is easy to see that

$$(R^{tr}\mathbb{R}[s]^l)^\perp = Ker\ R(\sigma). \tag{2}$$

Lemma 1 (*Duality Theorem*)**.** *Let R be a polynomial matrix of size $l \times q$. Then*

$$(Ker\ R(\sigma))^\perp = R^{tr}\mathbb{R}[s]^l = (Ker\ R(\sigma) \cap A(\mathbb{Z}_+^n)^q)^\perp.$$

Proof. Let E be either $C(\mathbb{Z}_+^n)^q$ or $A(\mathbb{Z}_+^n)^q$. We have to show that

$$(Ker\ R(\sigma) \cap E)^\perp = R^{tr}\mathbb{R}[s]^l.$$

By (1), the inclusion "$\supseteq$" is immediate. To show "$\subseteq$", take any $h \in \mathbb{R}[s]^q$ that does not belong to $R^{tr}\mathbb{R}[s]^l$. Then, by the cogenerator property of E, there exists a homomorphism

$$\mathbb{R}[s]^q / R^{tr}\mathbb{R}[s]^l \to E$$

taking the coset of h to a nonzero element. In other words, there is a homomorphism $\phi : \mathbb{R}[s]^q \to E$ that is zero everywhere on $R^{tr}\mathbb{R}[s]^l$, but not on h. Multiplying ϕ, if necessary, by some power $s^i = s_1^{i_1} \ldots s_r^{i_r}$, we may assume that $(\phi(h))(0) \neq 0$. Any homomorphism $\mathbb{R}[s]^q \to E$ is of the form $f \mapsto f^{tr}(\sigma)g$ with $g \in E$. So that there is $g \in E$ such that

$$\phi(f) = f^{tr}(\sigma)g \quad \forall f \in \mathbb{R}[s]^q.$$

For every $u \in \mathbb{R}[s]^l$, we have

$$0 = \phi(R^{tr}u) = u^{tr}(\sigma)R(\sigma)g.$$

It follows that $R(\sigma)g = 0$, and hence $g \in Ker\ R(\sigma) \cap E$. On the other hand,

$$0 \neq \phi(h)(0) = (h^{tr}(\sigma)g)(0) = \langle h, g \rangle,$$

that is, h is not orthogonal to $Ker\ R(\sigma) \cap E$.

The lemma is proved. □

One important consequence of the duality theorem is the Willems theorem. We shall carry out its proof, for the reader's convenience (and to make the text self-contained).

Lemma 2 (*Willems Theorem*)**.** *Let X be a subset of $C(\mathbb{Z}_+^n)^q$. For X to be the solution set of a linear constant coefficient partial difference equation it is necessary and sufficient that X be linear, shift-invariant and closed.*

Proof. Suppose that X is linear, shift-invariant and closed subspace in $C(\mathbb{Z}_+^n)^q$. Then $X^\perp$ is a submodule of $\mathbb{R}[s]^q$ and hence has the form $R^{tr}\mathbb{R}[s]^l$, where l is an integer and R is a polynomial matrix of size $l \times q$. By the duality theorem, we have

$$X^\perp = (Ker\ R(\sigma))^\perp.$$

This implies that

$$X^* = (Ker\ R(\sigma))^*.$$

Using the duality theorem and (2), we get

$$X \subseteq X^{\perp\perp} = (Ker\ R(\sigma))^{\perp\perp} = (R^{tr}\mathbb{R}[s]^l)^\perp = Ker\ R(\sigma).$$

Because X is closed and has the same continuous linear functionals as $Ker\ R(\sigma)$, by the Hahn–Banach theorem, we must have $X = Ker\ R(\sigma)$.

The necessity is obvious. □

Another important consequence of the duality theorem is the approximation theorem saying that the "A-solutions" of a linear constant coefficient partial difference equation are dense in the set of all solutions. This is the discrete-time analog of the Malgrange approximation theorem (see Theorem 7.14 in Hörmander [7]).

Lemma 3 (*Approximation Theorem*)**.** *Let R be a polynomial matrix. Then*

$$\overline{Ker\ R(\sigma) \cap A(\mathbb{Z}_+^n)^q} = Ker\ R(\sigma).$$

Proof. Let $l \times q$ be the size of R. By the duality theorem,

$$Ker\ R(\sigma)^\perp = (Ker\ R(\sigma) \cap A(\mathbb{Z}_+^n)^q)^\perp.$$

This yields

$$(Ker\ R(\sigma))^* = (Ker\ R(\sigma) \cap A(\mathbb{Z}_+^n)^q)^*.$$

If the lemma were false, by the Hahn–Banach theorem, we would have a nontrivial continuous linear functional on $Ker\ R(\sigma)$ that vanishes on $Ker\ R(\sigma) \cap A(\mathbb{Z}_+^n)^q$. But this is in contradiction with what we have obtained. □