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Transverse contraction criteria for existence, stability, and robustness of a limit cycle

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ABSTRACT

This paper derives a differential contraction condition for the existence of an orbitally-stable limit cycle in an autonomous system. This transverse contraction condition can be represented as a pointwise linear matrix inequality (LMI), thus allowing convex optimisation tools such as sum-of-squares programming to be used to search for certificates of the existence of a stable limit cycle. Many desirable properties of contracting dynamics are extended to this context, including the preservation of contraction under a broad class of interconnections. In addition, by introducing the concepts of differential dissipativity and transverse differential dissipativity, contraction and transverse contraction can be established for interconnected systems via LMI conditions on component subsystems.

1. Introduction

Dynamic systems with periodic solutions are important in many areas of engineering, including biologically-inspired robot locomotion, phase-locked loops, vortex shedding from aircraft wings, and combustion oscillations, to name just a few. Periodic behaviour is just as pervasive in biology [1].

The basic question we address in this paper is the following: when does an autonomous system of the form

$$\dot{x} = f(x) \tag{1}$$

have the property that all solutions starting from a particular set K converge asymptotically to a unique limit cycle? We will also address the question of when this property is preserved under various system interconnections. It is well known that periodic solutions of an autonomous differential equation can never be asymptotically stable. This is clear from the fact that two solutions initialised on the periodic orbit but offset in phase will never converge.

There is a long and distinguished history of research into limit cycles for nonlinear systems. For example, the famous result of Poincaré–Bendixson gives a very simple condition for planar systems, and an important generalisation to monotone cyclic feedback systems was published in [2]. For general systems, Birkhoff gave necessary conditions for periodic solutions in terms of the existence of particular "phase variables" [3], although these conditions imply the existence of *at least one* limit cycle, but give no insight into the number of limit cycles, or their stability.

In recent years, there has been substantial interest in using optimisation methods to search for "stability certificates" such as Lyapunov functions and barrier certificates [4–6]. Regions of attraction to periodic orbits have an interesting structure: they must be a continuous deformation of a torus: the Cartesian product of an open unit disc of dimension $n-1$, with a scalar circle coordinate [7]. In previous papers, the first author and others have extended the computational approach to limit cycle analysis using "transverse dynamics" and sum-of-squares programming [8–10]; however this method is not applicable when the system dynamics are uncertain, since uncertainty will generally change the location of the limit cycle in state space.

An alternative to Lyapunov methods is to search for a contraction metric [11,12]. For the purposes of robust stability analysis of equilibria, an important difference is that a Lyapunov function must generally be constructed about a known equilibrium, whereas a contraction metric implies the existence of a stable equilibrium indirectly. This is particularly useful if the equilibrium point may change location depending on the unknown dynamics.

Historically, basic convergence results on contracting systems can be traced back to the 1949 results of Lewis in terms of Finsler metrics [13], and results of Hartman [14] and Demidovich [15]. To our knowledge, contraction to limit cycles was first investigated using an identity metric by Borg [16], and Hartman and Olech [17].

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This line of analysis was extended by Leonov and colleagues to more general metrics [18] and attractors [19,20].

In this paper, we introduce *transverse contraction*, extending the results of [16–18] by exploiting generalised metrics and system combination properties as in [11]. We also introduce *differential dissipativity* and *transverse differential dissipativity*, as well as LMI conditions for each, giving a framework for optimisation-based analysis of complex interconnections of nonlinear systems.

2. Problem setup and preliminaries

We assume that $f : K \to \mathbb{R}^n$ in (1) is smooth and $x \in \mathbb{R}^n$, and that a unique solution of (1) exists. A set K is called *strictly forward invariant* under f if any solution of (1) starting with $x(0)$ in K is in the interior of K for all $t > 0$. A periodic solution $x^\star$ is one for which there exists some $T > 0$ such that $x^\star(t) = x^\star(t+T)$ for all t. Equilibria are trivially periodic for every T, but for oscillatory solutions – which are our main concern – there is some minimal time T such that the above holds and this is referred to as the *period*.

The *orbit* of a periodic solution is the set $\mathcal{X}^\star := \{x : x = x^\star(t)$ for some $t\}$. Note that while non-trivial periodic *solutions* cannot be asymptotically stable, their *orbits* can be, and in this case we say that the solution is *orbitally stable* (see, e.g., [21]). Define a *time reparametrisation* $\tau(t)$ as a smooth function $\tau : [0, \infty) \to [0, \infty)$ such that $\tau(t)$ is monotonically increasing and $\tau(t) \to \infty$ as $t \to \infty$.

A Finsler function [22] on a manifold M with a tangent bundle TM is a smooth function $V : TM \to \mathbb{R}$ satisfying *positive-definiteness*: for all $x \in M$, $V(x, \delta) > 0$ for $\delta \neq 0$ and $V(x, 0) = 0$, *homogeneity*: $V(x, \alpha\delta) = \alpha V(x, \delta)$ for $\alpha > 0$, and *convexity*: the Hessian matrix of V^2 with respect to δ is positive-definite for all x. In fact the homogeneity requirement can be relaxed, but we keep it here because it simplifies certain statements.

3. Contraction conditions for limit cycles

In this section we introduce a *transverse contraction* condition for an autonomous dynamical system (1). The condition is given in terms of a Finsler function $V(x, \delta)$. For most of this paper, we will assume a Riemannian metric $V(x, \delta) := \sqrt{\delta' M(x)\delta}$ where $M(x)$ is positive-definite for all x; however, the results of this section hold for more general Finsler metrics [22,13,23].

Definition 1. A system of the form (1) is said to be *transverse contracting* with rate $\lambda > 0$ on a set $K \subset M$ if there exists a Finsler function $V(x, \delta)$ satisfying

$$\frac{\partial V(x,\delta)}{\partial x} f(x) + \frac{\partial V(x,\delta)}{\partial \delta}\frac{\partial f(x)}{\partial x}\delta \leq -\lambda V(x,\delta), \tag{2}$$

for all $\delta \neq 0$ such that $\frac{\partial V}{\partial \delta} f(x) = 0$.

Note that the latter condition is a form of "transversality" of δ to the flow of the system. In the particular case that $V(x, \delta) := \sqrt{\delta' M(x)\delta}$, it is true if $\delta' M(x) f(x) = 0$, i.e. δ and $f(x)$ are orthogonal with respect to the metric $M(x)$. If transverse contraction is indicated without specifying a rate λ, then it is meant that the condition above holds for some $\lambda > 0$.

Theorem 1. *Let $K \subset \mathbb{R}^n$ be compact, smoothly path-connected, and strictly forward invariant. If (1) is transverse contracting on K with rate $\lambda > 0$ then for every two solutions x_1 and x_2 with initial conditions in K there exists time reparametrisations $\tau(t)$ such that $x_1(t) \to x_2(\tau(t))$ as $t \to \infty$. Furthermore, the convergence is exponential with rate λ.*

Proof. By definition, there exists a smooth path between any two points $x_1 \in K$ and $x_2 \in K$ that remains in K. Such a path can be considered as a smooth mapping $\gamma : [0, 1] \to K$ with $\gamma(0) = x_1$ and $\gamma(1) = x_2$. We assume that paths are parametrised so that $\frac{\partial\gamma(s)}{\partial s} \neq 0$ for all s.

Denote by $\Gamma(x_1, x_2)$ the set of all such smooth paths between x_1 and x_2 remaining in K and associate with each a length

$$L(\gamma) = \int_0^1 V\left(\gamma(s), \frac{\partial}{\partial s}\gamma(s)\right) ds$$

and consider the Riemann–Finsler distance between x_1 and x_2:

$$d(x_1, x_2) = \min_{\gamma \in \Gamma(x_1,x_2)} L(\gamma). \tag{3}$$

A minimizing curve γ is guaranteed to exist by the Hopf–Rinow theorem [22].

Let us consider a path parametrised both in s and time t : $\gamma(s, t)$, with the property that $\gamma(s, t_0)$ is the minimiser in (3) for two points $x_1(t_0)$ and $x_2(t_0)$. Now, let us introduce at every point $s \in [0, 1]$ and $t \geq t_0$ a smooth "speed scale" $\alpha(s, t) > 0$. That is, at each point $\gamma(s, t)$ we have

$$\frac{d}{dt}\gamma(s,t) = \alpha(s,t) f(\gamma(s,t))$$

with boundary conditions $\alpha(0, t) = 1$ and $\alpha(s, 0) = 1$. Now, by definition of the distance,

$$\frac{d}{dt} d(x_1(t), x_2(\tau(t))) \leq \int_0^1 \left[\frac{d}{dt} V\left(\gamma(s,t), \frac{\partial}{\partial s}\gamma(s,t)\right)\right] ds.$$

A sufficient condition for decrease of distance is decrease of the integrand pointwise:

$$\frac{d}{dt} V\left(\gamma(s,t), \frac{\partial}{\partial s}\gamma(s,t)\right) = \frac{\partial V(x,\delta)}{\partial x}\dot{x} + \frac{\partial V(x,\delta)}{\partial \delta}\dot{\delta} < 0$$

evaluated at $x = \gamma(s, t)$ and $\delta = \frac{\partial}{\partial s}\gamma(s, t)$, i.e. with

$$\dot{x} = \alpha(s,t) f(\gamma(s,t)),$$
$$\dot{\delta} = \frac{d}{dt}\frac{\partial}{\partial s}\gamma(s,t) = \frac{\partial}{\partial s}(\alpha(s,t) f(\gamma(s,t)))$$
$$= \frac{\partial \alpha}{\partial s} f(\gamma(s,t)) + \alpha(s,t)\frac{\partial f}{\partial x}\frac{\partial \gamma}{\partial s}.$$

Hence decrease of distance can be guaranteed if

$$\frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial \delta}\left(z f(x) + \frac{\partial f}{\partial x}\delta\right) < 0 \tag{4}$$

where the above has been normalised by $\alpha(s, t) > 0$, which does not affect the sign of the inequality, and where $z = \frac{1}{\alpha(s,t)}\frac{\partial \alpha}{\partial s}$ is a scalar. Considering z as a "control input" which can be freely chosen, it is obviously easy to achieve the above inequality by choice of z except when $\frac{\partial V}{\partial \delta} f(x) \neq 0$, for which the system must be "naturally" contracting, which is the definition of transverse contraction (2).

The time reparametrisation $\tau(t)$ is constructed by integrating $\alpha(1, t)$ with respect to time. The existence of a monotonic reparametrisation is straightforward to show and we omit the details. □

Remark 1. Stability under time reparametrisation is sometimes referred to as *Zhukovsky stability* and has been used previously to study limit cycle stability, see e.g. [24,18,25,26,10], and apparently goes back to Poincaré in its essential argument [21]. It is known that systems satisfying such a property have limit cycles [25], but we include a proof here since it is straightforward using the constructions in the previous theorem.

Theorem 2. *If the conditions of Theorem 1 are satisfied, then all solutions starting with $x(0) \in K$ converge to a unique limit cycle. Furthermore, convergence is exponential with rate λ and has the property of asymptotic phase: i.e. for any initial conditions $x_1(0), x_2(0)$, there exists a fixed τ such that $x_1(t) \to x_2(t + \tau)$ exponentially.*