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On the stability of receding horizon control for continuous-time stochastic systems

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ABSTRACT

We study the stability of receding horizon control for continuous-time non-linear stochastic differential equations. We illustrate the results with a simulation example in which we employ receding horizon control to design an investment strategy to repay a debt.

1. Introduction

In Receding Horizon Control (RHC), the control action, at each time t in $[0, \infty)$, is derived from the solution of an optimal control problem defined over a finite future horizon $[t, t+T]$. The RHC strategy establishes a feedback law which, under certain conditions, can ensure asymptotic stability of the controlled system. This control strategy has been successfully developed over the last twenty years for systems described by deterministic equations. In this context RHC is also well known as Model Predictive Control (MPC) and has proven to be very successful in dealing with non-linear and constrained systems, see e.g. [1–3]. The extension of RHC from deterministic to stochastic systems is the objective of current research. RHC schemes for the control of discrete-time stochastic systems have been proposed recently in [4–7].

In this note, we discuss RHC for systems described by continuous-time non-linear stochastic differential equations (SDEs). In order to study the stability of RHC for continuous-time SDEs, we formulate conditions under which the value function of the associated finite-time optimal control problem can be used as Lyapunov function for the RHC scheme. This is a well established approach for studying the stability of RHC schemes, which here is extended using Lyapunov criteria for stochastic dynamical systems [8]. We illustrate this contribution with a simple example of an optimal investment problem. Optimal investment problems are well suited to be tackled by stochastic control methods, see e.g. [9, 10]. In our example, we design an investment strategy to repay a debt. Having negative wealth due to an initial debt, the investor has the option to increase his/her current debt in order to buy a risky asset. The asymptotic stability of the adopted RHC scheme guarantees that the wealth of the investor tends to zero, so that the initial debt is eventually repaid.

2. Problem statement

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space equipped with the natural filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ generated by a standard Wiener process $W : [0,\infty) \times \Omega \to \mathbb{R}^d$ on it. We consider a controlled time-homogeneous SDE for a process $X : [0,\infty)\times\Omega \to \mathbb{R}^n$,

$$dX_t^{0,x_0,u} = b(X_t^{0,x_0,u}, u_t)dt + \sigma(X_t^{0,x_0,u}, u_t)dW_t, \qquad (1)$$
$$X_0^{0,x_0,u} = x_0,$$

where $x_0 \in \mathbb{R}^n$; $b : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $\sigma : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{n\times d}$ are continuous functions and satisfy

$$|b(x,u)| + |\sigma(x,u)| \leqslant C(1+|x|+|u|), \quad \forall (x,u) \in \mathbb{R}^n \times U, \ \text{(linear growth)},$$

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and

$$|b(x,u)-b(y,u)|+|\sigma(x,u)-\sigma(y,u)| \leqslant C|x-y|, \quad \forall(x,y,u)\in\mathbb{R}^n\times\mathbb{R}^n\times U, \text{ (Lipschitz)},$$

for some constant $C > 0$; and $u_{(\cdot)}$ is an admissible control process

$$u_{(\cdot)} \in \mathcal{U} := \left\{ u : [0,\infty)\times\Omega \to U : \text{ progressively measurable and } \mathbb{E}\int_0^\infty |u_t(\omega)|^2 dt < \infty \right\},$$

with the set $U \subset \mathbb{R}^m$ compact. Here the superscripts of $X^{0,x_0,u}$ mean that the initial value of the process at time 0 is x_0 and the involved control process is $u_{(\cdot)}$. Under above conditions the SDE (1) has a unique adapted continuous square integrable solution X_t, $t \geqslant 0$. In this paper we are concerned with the conditions under which there exists a control process that drives the stochastic system X to the origin $0 \in \mathbb{R}^n$ and guarantees asymptotic stability of the controlled process. Here, the following definition of stability is adopted [8]:

Definition 2.1. Given a stochastic continuous-time process $X : \mathbb{R}_+ \times \Omega \to \mathbb{R}^n$, where $\mathbb{R}_+ := [0,\infty)$, with $X_0 = x_0 \in \mathbb{R}^n$

(S1) The origin is stable almost surely if and only if, for any $\rho > 0$, $\epsilon > 0$, there is a $\delta(\rho,\epsilon) > 0$ such that, if $|x_0| \leqslant \delta(\rho,\epsilon)$,

$$\mathbb{P}\left[\sup_{t\in\mathbb{R}_+}|X_t| \geqslant \epsilon\right] \leqslant \rho.$$

(S1′) An equivalent definition to (S1) is: Let $h(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ be a scalar-valued, nondecreasing, and continuous function of $|x|$. Let $h(0) = 0$, $h(r) > 0$ for $r \neq 0$. Then the origin is stable almost surely if and only if, for any $\rho > 0$, $\lambda > 0$, there is a $\delta(\rho,\lambda) > 0$ such that, for $|x_0| \leqslant \delta(\rho,\lambda)$,

$$\mathbb{P}\left[\sup_{t\in\mathbb{R}_+} h(|X_t|) \geqslant \lambda\right] \leqslant \rho.$$

(S2) The origin is asymptotically stable almost surely if and only if it is stable a.s., and $X_t \to 0$ a.s. for all x_0 in some neighbourhood R of the origin. If $R = \mathbb{R}^n$ then we add 'in the large'.

3. Main results

Let $T > 0$. As a preliminary step we consider the SDE for $X : [t,T]\times\Omega \to \mathbb{R}^n$ starting from the point $x \in \mathbb{R}^n$ at the time $t \in [0,T]$

$$dX_s^{t,x,u} = b(X_s^{t,x,u}, u_s)ds + \sigma(X_s^{t,x,u}, u_s)dW_s, \qquad (2)$$
$$X_t^{t,x,u} = x.$$

Let $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_+$ and $g : \mathbb{R}^n \to \mathbb{R}_+$ be continuous nonnegative functions of polynomial growth, that is,

$$|f(x,u)| \leqslant C(1+|x|^p+|u|^q), \quad \forall(x,u)\in\mathbb{R}^n\times U,$$

and

$$|g(x)| \leqslant C(1+|x|^p), \quad \forall x\in\mathbb{R}^n,$$

for some constant $C > 0$ and some $p, q \geqslant 1$. Now we consider the problem of minimizing the following cost functional, $\forall(t,x) \in [0,T]\times\mathbb{R}^n$,

$$J[t,x;T;u_{(\cdot)}] := \mathbb{E}\left[\int_t^T f(X_s^{t,x,u}, u_s)ds + g(X_T^{t,x,u})\right] \qquad (3)$$

over the set $\mathcal{U}$ of admissible control processes. We define the value function as

$$v(t,x;T) := \inf_{u_{(\cdot)}\in\mathcal{U}} J[t,x;T;u_{(\cdot)}] = \inf_{u_{(\cdot)}\in\mathcal{U}} \mathbb{E}\left[\int_t^T f(X_s^{t,x,u}, u_s(t,x;T))ds + g(X_T^{t,x,u})\right] \qquad (4)$$

and denote $u_s^*(t,x;T)$, $t \leqslant s \leqslant T$, the optimal control process if it exists. In particular, when $t = 0$ we denote $V(x;T) := v(0,x;T)$.

Standard stochastic optimal control theories (see, for instance, [11,12]) about the controlled SDE (1) tell us that the Hamilton–Jacobi–Bellman (HJB) equation for the value function $v(\cdot,\cdot;T)$ is, $\forall(t,x)\in[0,T]\times\mathbb{R}^n$,

$$-\partial_t v(t,x;T) = \inf_{u\in U}\left[\frac{1}{2}\mathrm{tr}[\sigma\sigma^*(x,u)D^2v(t,x;T)] + \langle b(x,u), Dv(t,x;T)\rangle + f(x,u)\right], \qquad (5)$$

$$v(T,x;T) = g(x).$$

Hereafter we use the notations

$$\partial_t v := \frac{\partial v}{\partial t}, \qquad Dv = \begin{pmatrix} \frac{\partial v}{\partial x_1} \\ \vdots \\ \frac{\partial v}{\partial x_n}\end{pmatrix}, \quad \text{and}$$

$$D^2 v = \begin{pmatrix} \frac{\partial^2 v}{\partial x_1^2} & \frac{\partial^2 v}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 v}{\partial x_1\partial x_n} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial^2 v}{\partial x_n\partial x_1} & \frac{\partial^2 v}{\partial x_n\partial x_2} & \cdots & \frac{\partial^2 v}{\partial x_n^2}\end{pmatrix}.$$

Suppose this HJB equation has a unique classical solution (see, e.g. [11] for conditions guaranteeing the existence and uniqueness.) and that the infimum in the equation is attained by $\tilde{u}(t,x;T)$ for every $(t,x)\in[0,T]\times\mathbb{R}^n$, i.e.,

$$-\partial_t v(t,x;T) = \frac{1}{2}\mathrm{tr}[\sigma\sigma^*(x,\tilde{u}(t,x;T))D^2v(t,x;T)] + \langle b(x,\tilde{u}(t,x;T)), Dv(t,x;T)\rangle + f(x,\tilde{u}(t,x;T)),$$

then we construct the optimal control process for the SDE (2) as

$$u_s^*(t,x;T) := \tilde{u}(s, X_s^{t,x,\tilde{u}}; T), \quad t \leqslant s \leqslant T. \qquad (6)$$

As a consequence, the value function turns out to be

$$v(t,x;T) = \mathbb{E}\left[\int_t^T f(X_s^{t,x,u^*}, u_s^*)ds + g(X_T^{t,x,u^*})\right] = \mathbb{E}\left[\int_t^T f(X_s^{t,x,\tilde{u}}, \tilde{u}(s, X_s^{t,x,\tilde{u}};T))ds + g(X_T^{t,x,\tilde{u}})\right].$$

Now, for all the states $x\in\mathbb{R}^n$ all the time, we apply the specifically designed feedback law

$$u^c(x;T) := \tilde{u}(0,x;T) \qquad (7)$$

to the stochastic system (1). In other words, for the state X_t^{0,x_0,u^c}, at any time $t \geqslant 0$, we apply only the initial optimal control

$$u^c(X_t^{0,x_0,u^c};T) = \tilde{u}(0, X_t^{0,x_0,u^c};T) = u_0^*(0, X_t^{0,x_0,u^c};T)$$

to the system. In particular, for $t = 0$, $u^c(X_0^{0,x_0,u^c};T) = u^c(x_0;T) = \tilde{u}(0,x_0;T)$. We require $u^c(\cdot;T)$ to be continuous and call it the continuous receding horizon control process with the receding horizon T for the controlled SDE (1).